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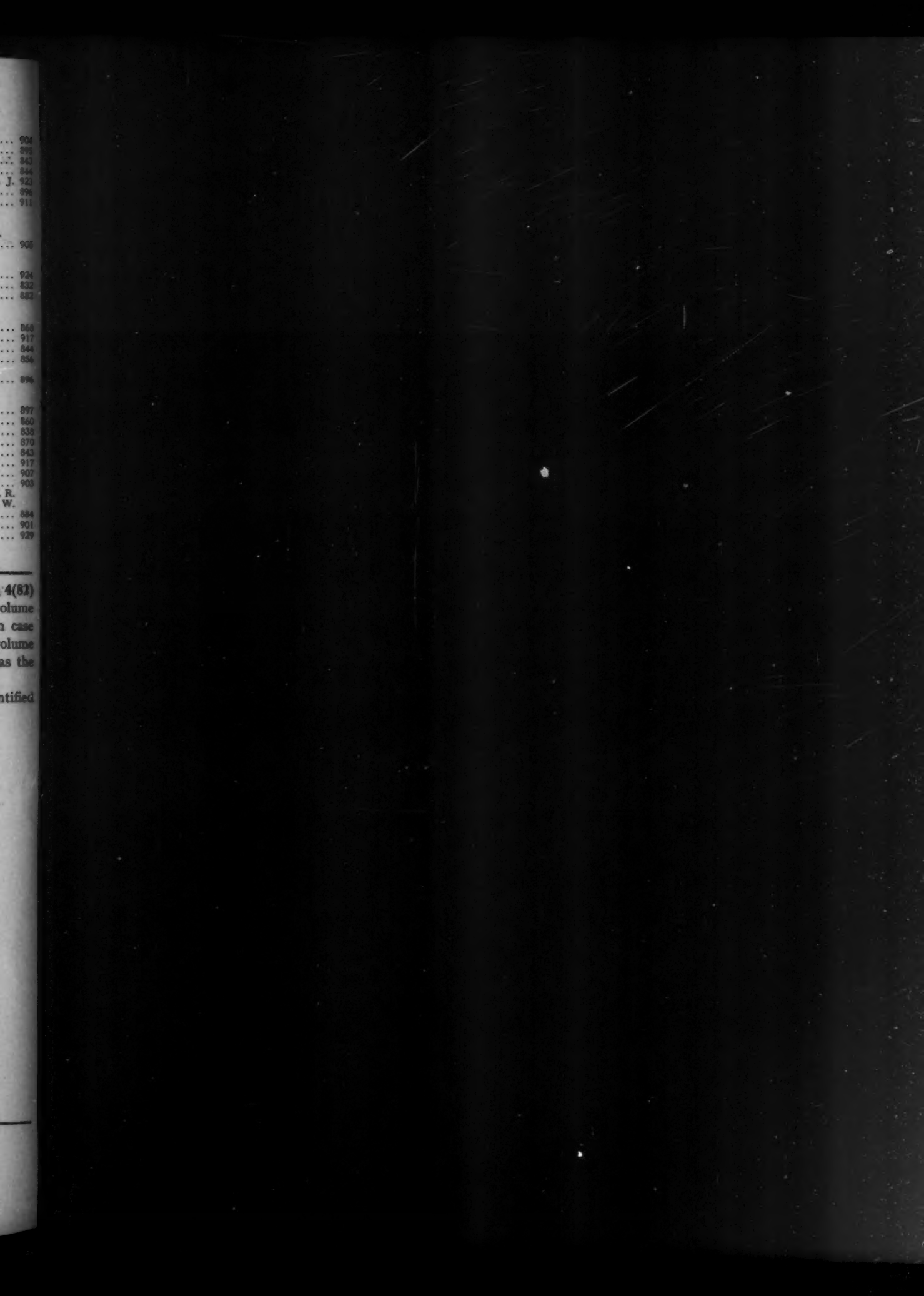

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FOUNDATIONS, THEORY OF SETS, LOGIC

Takeuti, Gaisi. Remark on my paper: On Skolem's theorem. J. Math. Soc. Japan **9** (1957), 192–194.

Verf. erweitert seinen Beweis [dasselbe J. **9** (1957), 71–76; MR **19**, 4], dass ein Axiomensystem der Mengenlehre konsistent bleibt bei Hinzufügung von Axiomen, die die Abzählbarkeit jeder Menge ausdrücken, auf die Axiomatisierungen im Gentzenschen Kalkül *LK*.

P. Lorenzen (Kiel).

Kozlova, Z. I. On covering of sets. II. Izv. Akad. Nauk SSSR. Ser. Mat. **21** (1957), 349–370. (Russian)

The paper is a continuation of previous results [same Izv. **19** (1955), 125–132; MR **16**, 909; the same terminology is used]. For every sequence $n_1, \cdots, n_k$ of ordinals $<\omega$ let $E_{n_1,\cdots,n_k}$ be a set; the corresponding A-operation yields the set $A(\{E_{n_1,\cdots,n_k}\})=\bigcup_\eta \bigcap_k E_{n_1,\cdots,n_k}$ $(k<\omega, \eta \in J)$, J denoting the set of all ω-sequences of ordinals $<\omega$; J is the Baire space where for any $\eta=(n_1, n_2, \cdots)$ the neighbourhoods of η are of the form $\delta(n_1, \cdots, n_k)=\{(n_1, n_2, \cdots, n_k, n_{k+1}, \cdots)\}$. For any system N of sets of ordinals $<\omega$, let $\phi_N(E_n)=\bigcup_{\eta\in N} \bigcap_{n<\eta} E_n$ for every sequence of sets E_n; ϕ_N is the δs-operation with basis N. An A-operation is also such a ϕ_N-operation: it is sufficient to consider a (1-1)-mapping ν of all finite sequences $n_1, \cdots, n_k$ onto the set $I\omega$ of ordinals $<\omega$ and put $\nu J=\{\nu(n_1, n_2, \cdots)=(\nu(n_1), \nu(n_1, n_2), \cdots) \mid (n_1, n_2, \cdots)\in J\}$; then $A(\{E_{n_1,\cdots,n_k}\})=\phi_{\nu J}(\{E_n\})$ $(n<\omega)$. For any basis N, one has $N'=\nu N\subseteq J$; and the terminology concerning abstract spaces is used; e.g., N' is nowhere dense in J. In particular, any subset X of $I\omega$ might be considered in the increasing form $\overrightarrow{X}$; one puts $\overrightarrow{N}=\{\overrightarrow{s} \mid s\in N\}$. Let N^{***} denote the set of all the elements of N each of which is the union of $>\aleph_0$ other elements of N; let N_* denote the system of all the elements of N, each of which is the union of other elements of N having a non-compact closure in J; for any ordinal α let N_α be composed of all $x\in N$ which are non-dispersed of class $\leqq\alpha$. N and S (=system of sets) are in completely regular relation provided $\phi_N(S)$ is a $\sigma\delta$-system and $\phi_M(S)\subseteq\phi_N(S)$ $(M\in K$; K denotes all the finite intersections of the systems N, $N^k=\{x \mid k\in x\in N\}$ $(k<\omega)$. A class $\Lambda=\{\beta(x)\}$ of ordinals $\leqq\Omega$ is said to be completely regular with respect to a system S of sets provided 1) $[\beta(x)=\Omega]=\{x \mid \beta(x)=\Omega\}\in S$ for every $\beta(x)\in\Lambda$; 2) every element of S is representable in the form as in 1); 3) for every $\beta_1(x)$, $\beta_2(x)\in\Lambda$ the set $[\beta_1(x)\geqq\beta_2(x)]\in S$; 4) if E_1, $E_2\in S$ there are $\beta_1(x)$, $\beta_2(x)\in\Lambda$ satisfying $[\beta_1(x)=\beta_2(x)]=E_1\cap E_2$, $E_i=[\beta_i(x)=\Omega]$ $(i=1, 2)$. As a synthesis of various other results the following general "covering theorem" is proved (the accent is on the existence of B-sets!).

Let N be a rigid basis of a δs-operation ϕ_N, S a class of sets invariant with respect to ϕ_N and in completely regular relation with N, Λ a system of indices in completely regular relation with S, U a point property, and ϕ_M a δs-operation selecting just the points with U-property such that $\phi_M S\subseteq\phi_N S$ and $\phi_{M^n}S\subseteq\phi_N S$ $(n<\omega)$; then for every sequence $E_n\in S$ such that no point of $\phi_N\{E_n\}$ has the U-property there exists a sequence H_n of BS-sets such that $H_n\supseteq E_n$ and that no point of $\phi_N(H_n)$ has the U-property (Th. I). For every class S of sets one has $\phi_{N^{***}}S\cup\phi_{N^{***}_*}S\subseteq\phi_N S$ $(n<\omega)$ (Th. III). If moreover S and N are in completely regular relation, $\phi_N S$ contains both $\phi_{N_\alpha}S\cup\phi_{N_\alpha{}^n}S$, $\phi_{N_*}S\cup\phi_{N_*{}^n}S$ for every $n<\omega$ (Theorems IV and V respectively). There is also an analogous theorem (Th. VI) dealing with $N_*^{(\alpha)}$, $N_{*^n}^{(\alpha)}$ denoting the set of all the elements of N which are non-dispersed of index $\leqq\alpha$ and the closure of which is compact. The paper contains several corollaries to Theorems I–VI; e.g., Cor. 4 of Th. V says that for every sequence E_n of A-sets, such that every point of $\phi_N(\{E_n\})$ is defined by means of a compact system of elements of N, there exists a sequence H_n of B-sets such that $H_n\supseteq E_n$ and that every point of $\phi_N(\{H_n\})$ is definable by a set of elements of N with compact closure.

Đ. Kurepa (Zagreb).

★ **Sierpiński, Wacław. Hypothèse du continu. 2nd ed.** Chelsea Publishing Company, New York, N. Y., 1956. xvii+274 pp. $4.95.

Photographic reproduction of this well-known work [Warszawa-Lwów, 1934], plus sixteen later papers on the same subject. As the original was not reviewed in MR, some brief remarks may be in order now. The author considers over one hundred propositions, dealing for the most part with linear sets. A dozen are proved equivalent to the continuum hypothesis; the others are derived as consequences, and various implications among them are established. Chapter I: Propositions équivalents à l'hypothèse du continu [propositions P_1-P_{11}]; II: L'ensemble de M. Lusin [propositions C_1-C_{24}]; III: Applications aux relations entre catégorie et mesure [C_{25}-C_{47}]; IV: Autres conséquences de l'hypothèse du continu [C_{48}-C_{80}]; V: Hypothèse des alephs inaccessibles [C_{81}-C_{82}]; VI: Hypothèse du continu et les exemples effectifs; VII: Hypothèse du continu généralisée. The later papers include a correction needed to fill a gap in the proof on p. 13, and the paper in which the axiom of choice is derived from the generalized continuum hypothesis [Fund. Math. **34** (1947), 1–5; MR **8**, 506].

L. Gillman (Lafayette, Ind.).

Eyraud, Henri. Le théorème de l'ordinal limite. II. Ann. Univ. Lyon. Sect. A. (3) **19** (1956), 7–12.

[For Part I see same Ann. (3) **18** (1955), 5–14; MR **18**, 551.] Yet another attempt by this author to prove the continuum hypothesis. *L. Gillman* (Lafayette, Ind.).

Fraïssé, Roland. Application des γ-opérateurs au calcul logique du premier échelon. Z. Math. Logik Grundlagen Math. **2** (1956), 76–92.

Mise au point de résultats d'une Note antérieure [C. R. Acad. Sci. Paris **240** (1955), 2191–2193; MR **16**, 1006]. L'A. définit les "formules logiques du premier et du second échelon", et différencie les individus d'une formule en se servant d'indices, ce qui facilite l'exposé. La

caractéristique d'une formule ϕ est désignée par $\gamma=\binom{n}{p}$, n étant le degré apparent et p le nombre des individus liés de ϕ. On convient que $\gamma\leqq\gamma'$ équivaut à $n\leqq n'$, $p\leqq p'$. Les formules sont ordonnées par l'inclusion.

Soit μ_ϕ le nombre maximal des indices libres de ϕ, ϕ pouvant avoir un prédicat à m places vides; soit $m'\geqq\mu_\phi$; associons à chaque m-relation R une relation R' par $R'(a_1\cdots a_{m'})=\phi(R; a_1\cdots a_{m'})$. On a ainsi un opérateur $\Pi_\phi^{mm'}$ d'espèce $(m\to m')$ dit associé à ϕ. "Pour qu'un opérateur P d'espèce $(m\to m')$ soit un γ-opérateur, il faut et il suffit qu'il existe une formule ϕ de caractéristique $\gamma'\leqq\gamma$ telle que $P=\Pi_\phi^{mm'}$ (no. 4.1.Th.). Le cas $m'=0$ peut être exprimé en forme de la Proposition 5.2 que voici: "Pour qu'une classe Γ de relations soit une réunion de classes d'équivalence définies par la parenté $\mathfrak{L}$, il faut et suffit qu'il existe une formule ϕ de caractéristique $\gamma'\leqq\gamma$, dont les individus soient tous liés, et qui soit égale à $+$ (ou qui soit "vérifiée") pour les seules relations de Γ."

A partir de cela l'A. donne une interprétation en logique mathématique des parentés définies dans travaux antérieures [Thèse, Univ. de Paris, 1953; MR **15**, 296; et l'oeuvre cité ci-dessus]. *Đ. Kurepa* (Zagreb).

Smullyan, Raymond M. Languages in which self reference is possible. J. Symb. Logic **22** (1957), 55–67.

Verf. entwickelt eine Methode, die die "Arithmetisierung der Syntax" sehr vereinfacht. Der Grundgedanke besteht darin, die Arithmetisierung des Substitutionsprozesses zu vermeiden und statt dessen mit der Arithmetisierung der Verkettung auszukommen. Dies ist möglich, wenn man etwa die Arithmetik aufbaut auf Negatkonjunktion $\downarrow$, Mengenabstraktion $\alpha(F)$ und Gleichheit $=$. Als Variable α, β, $\cdots$ werden x, x', x'', $\cdots$, als Konstanten N, $\cdots$ werden 1, 11, 111, $\cdots$ genommen. Terme werden mithilfe von $\alpha\cdot\beta$ und $\alpha\llcorner\beta$ (statt α^β) gebildet. Statt $N\in\alpha(F)$ wird $\alpha(F)N$ geschrieben. Die Gödelnummer $\overline{E}$ eines Ausdrucks E wird so definiert: man ersetze in E die Zeichen x, $'$, $(,)$, $\cdot$, $\llcorner$, $=$, $\downarrow$, 1 durch 1, 2, $\cdots$, 9; das Resultat bezeichnet dezimal eine Zahl $g_0(E)$. Es sei $\overline{E}=g_0(E)+1$. Dies ist so eingerichtet, dass $\overline{E}\,\overline{\overline{E}}=\overline{E}\cdot 10^{\overline{E}}$, und das ist alles, was an "Arithmetisierung" gebraucht wird. Dass die Menge der falschen Aussagen E nicht im System definierbar ist, d.h. dass für keine Menge $\alpha(H)$ gilt

(1) $\alpha(H)\overline{E}$ wahr $\Leftrightarrow E$ falsch (für beliebige Aussagen E),

folgt sofort daraus, dass für die Formel H^*, die aus H bei Ersetzung von α durch $\alpha\cdot 10^\alpha$ (genauer: $\alpha\cdot 1111111111\llcorner\alpha$) entsteht, gilt

(2) $\alpha(H^*)\,\overline{P}$ wahr $\Leftrightarrow\alpha(H)\overline{P\overline{P}}$ wahr (für beliebige Mengen P);

(1) und (2) ergeben nämlich

(3) $\alpha(H^*)\overline{\alpha(H^*)}$ wahr $\Leftrightarrow\alpha(H^*)\overline{\alpha(H^*)}$ falsch.

Verf. stellt den Gedankengang in einleitenden Abschnitten für semantische Systeme im allgemeinen dar.

P. Lorenzen (Kiel).

Bočvar, D. A. On paradoxes and the extended calculus of predicates. Mat. Sb. N. S. **42(84)** (1957), 3–10. (Russian)

In a previous paper [Mat. Sb. N.S. **15(57)** (1944), 369–384; MR **7**, 46; see also Szmielew, J. Symb. Logic **11** (1946), 129] the author proposed an extended predicate calculus K_0 without any restrictions of type, such that, however, definition of a predicate P by an axiom of the form

(1) $P(x)\rightleftarrows\mathfrak{A}$,

where $\mathfrak{A}$ is a formula containing x as free variable, was not allowed. Such a K_0 was shown [loc. cit.] to be consistent. In the present paper he proposes some extensions of K_0. In the first of these, called $\mathfrak{K}_0$, there is added to the fundamental relation of predication (i.e., application of a predicate ϕ to an argument x to form $\phi(x)$) a second relation of groundedness expressed by $\phi\langle x\rangle$, with the axiom scheme

(α) $\phi\langle x\rangle\rightleftarrows\phi(x)\vee(E\psi)(\phi(\psi)\,\&\,\psi\langle x\rangle)$.

He then proposes a system $\mathfrak{K}$ formed by replacing each of these relations by a pair of relations, one weaker and one stronger; thus $\phi(x)$ is replaced by two relations x ex ϕ and ϕ in x, and $\phi\langle x\rangle$ by x e ϕ and ϕ i x. Besides the axiom scheme (α) for both weaker and stronger forms, $\mathfrak{K}$ has seven axiom schemes. The first two of these state that the first relation of each pair is stronger than the second. If we introduce the abbreviations

$$\mathfrak{R}\phi\equiv(\psi)(\psi\text{ e }\phi\to\neg(\psi\text{ e }\psi)),$$
$$\mathfrak{M}\phi\equiv(\psi)(\phi\text{ i }\psi\to\neg(\psi\text{ i }\psi)),$$
$$\mathfrak{S}\phi\equiv(x)(\psi\text{ in }x\to x\text{ ex }\phi),$$

the remaining axiom schemes are:

(III) $\psi\text{ ex }\phi\to\mathfrak{R}\psi$;

(IV) $\mathfrak{M}\psi\to(\phi\text{ in }\psi\to\psi\text{ ex }\phi)$;

(V) $(\dot{x})(\dot{x}\text{ex }\psi\to x\text{ ex }\phi)\,\&\,y\text{ ex }\phi\to(\psi\text{ in }y\to y\text{ ex }\psi)$;

(VI) $\mathfrak{S}\phi\,\&\,\mathfrak{S}\psi\,\&\,(x)(\phi\text{ in }x\rightleftarrows\psi\text{ in }x)\to\phi=\psi$;

and (VII) the definition (1) is allowed with 'ϕ in x' on the left provided $\mathfrak{A}$ is formed from ex alone without using in, i, e (of course with the use of usual logical notions). In this system it is shown that the Russell paradox as usually given does not arise. The formulation of a theory of sets on such a basis is promised for a separate paper. {Certain liberties with the author's notation have been taken; also only univariate functions are considered here.}

H. B. Curry (University Park, Pa.).

Sikorski, R. A theorem on non-classical functional calculi. Bull. Acad. Polon. Sci. Cl. III. **4** (1956), 649–650.

This note is a supplement to a paper of Rasiowa and Sikorski [Fund. Math. **40** (1953), 62–95; MR **15**, 668]. The question, whether the space $X_\lambda(X_\varkappa, X_\varpi)$ constructed there is metrizable, was left unanswered. It is announced that the answer is in the affirmative; proofs are to appear later. A lemma is the following: if X is a topological space with an enumerable open basis and of cardinality equal to or less than that of the reals, then X is an open image of a set of irrational numbers. *E. E. Floyd.*

Halmos, Paul R. Algebraic logic. IV. Equality in polyadic algebras. Trans. Amer. Math. Soc. **86** (1957), 1–27.

This paper is one of a series [Compositio Math. **12** (1956), 217–249; Fund. Math. **43** (1956), 255–325; Trans. Amer. Math. Soc. **83** (1956), 430–470; MR **17**, 1172; **19**, 112, 113] concerned with an algebraic version of the predicate calculus. The algebraic structures which arise in the process are called polyadic algebras and the present paper is concerned with the notion of equality in such algebras. It has been shown by B. A. Galler [Proc. Amer. Math. Soc. **8** (1957), 176–183; MR **19**, 113] that the theory of polyadic algebras with equality is essentially equivalent to the theory of cylindric algebras of Tarski, Thomson and Henkin, and the same is proved again in the present paper. Accordingly, a completeness theorem for locally finite

polyadic algebras of infinite degree with equality, which is proved in the present paper, should be essentially equivalent to Henkin's completeness theorem for cylindric algebras. It is shown further that while no relation of equality need exist in a given polyadic algebra, such an algebra can always be embedded in one with equality. Finally, the logical theory of descriptions is considered within the algebraic setting of the present paper.

A. Robinson (Jerusalem).

Kanger, Stig. The morning star paradox. Theoria, Lund **23** (1957), 1–11.

The essence of the morning star paradox [see, e.g., Quine, J. Symb. Logic **12** (1947), 43–48, p. 47; MR 8, 557] is that from the premises

$$\vdash N(x=x), \quad \vdash x=y_{\blacksquare}\supset_{\blacksquare}Fx\supset Fy,$$

where 'N' indicates necessity and 'F' is a free variable for a unary function, one can deduce

$$\vdash x=y_{\blacksquare}\supset_{\blacksquare}N(x=y);$$

this is incompatible with the existence of objects which are equal but not necessarily equal. Kanger gives a semantical argument to the effect that the second premise is not acceptable without restrictions on F, goes on to discuss views of Quine and Carnap on this subject, and finally concludes that intensional entities are not necessary for explanation of this and a number of similar situations (belief-statements, etc.). The argument is suggestive; but the reviewer is unable to follow it in detail. In fact the author's semantical argument requires that a "primary evaluation", which is only defined relative to a domain, should have a certain property for every domain; this point seems to need clarification. The reviewer suggests as addition to the rather full bibliography the paper by F. B. Fitch [Philos. Sci. **16** (1949), 137–141]. *H. B. Curry* (University Park, Pa.).

Hoering, W. Frege und die Schaltalgebra. Arch. Math. Logik Grundlagenforsch. **3** (1957), 125–126.

ALGEBRA

Yaqub, Adil. On the identities of certain algebras. Proc. Amer. Math. Soc. **8** (1957), 522–524.

A universal algebra $\mathfrak{A}$ consists of a set A together with a family of operations ρ: A^n into A, of various ranks $n\geqq 0$. $\mathfrak{A}$ is strictly functionally complete if every function f: A^m into A, arbitrary m, can be expressed by composition of the operations ρ of ranks $n>0$. Identity $\phi\equiv\psi$ holds between two formal composites ϕ and ψ of the operations ρ in case ϕ and ψ define the same function. The main theorem states that if $\mathfrak{A}$ is strictly functionally complete, and if A is finite and has more than one element, then all identities are formal consequences of some finite subset.

R. C. Lyndon (Ann Arbor, Mich.).

Combinatorial Analysis

Gilbert, E. N. Knots and classes of ménage permutations. Scripta Math. **22** (1956), 228–233 (1957).

The reduced problème des ménages, which asks for the number of permutations discordant with the two permutations $123\cdots n$ and $234\cdots n1$, arose from an attempt to solve a problem in the theory of knots posed by P. G. Tait. Actually Tait's problem called for the enumeration of these ménage permutations with the additional requirement that two permutations which are transforms of each other for any cyclic permutation are taken as the same. If $T(n)$ is the number of Tait's permutations, then the answer given here is

$$T(n)=\frac{1}{n}\sum_{d\mid n}\varphi(n/d)(n/d)^d U_d(1-d/n)$$

with the sum over all divisors d of n, $\varphi(d)$ the Euler totient function and $U_n(t)$ the "hit" polynomial for the reduced ménage problem. For $n=3(1)9$, the numbers $T(n)$ are 1, 2, 5, 20, 87, 616, 4843, and 44128. *J. Riordan.*

Riordan, John. The numbers of labeled colored and chromatic trees. Acta. Math. **97** (1957), 211–225.

The author determines enumerating functions for the number of trees. The trees are composed of two sets of elements, namely points and lines. The elements may all be unmarked; or a subset of one of the sets of elements containing a given number of these elements may be marked, each mark differing from every other mark; or finally each member of one of the sets of elements of the tree may be assigned a color out of a given store of colors, either the total number of colored elements or the total number of distinct colors being predetermined. For each of these types of marking, the marked elements may be the points or the lines, and the trees may be unoriented or oriented and unrooted or rooted. The treatment is thus general and inclusive. The methods used are successful adaptations of Pólya's well-known methods [Acta Math. **68** (1937), 145–254].

The following are two specimens of the results obtained.

The number r_{pm} of rooted trees with p points, m of which have distinct labels, are completely determined by the enumerator identity

$$\sum r_{pm}x^p y^p/m!=r(x,y)=$$
$$x(1+y)\exp[r(x,y)+r(x^2,0)/2+\cdots+r(x^k,0)/k+\cdots],$$

The number $q_p(c)$ of rooted trees with p points, each of which may be colored with any of c colors, is completely determined by the enumerator identity

$$\sum x^p q_p(c)=q(x;c)$$
$$=cx\exp[q(x;c)+q(x^2;c)/2+\cdots+q(x^k;c)/k+\cdots].$$

G. A. Dirac (Hamburg).

See also: Linear Algebra: Whittle.

Linear Algebra

Tatarkiewicz, Krzysztof. Sur l'orthogonalité généralisée des matrices propres. Ann. Univ. Mariae Curie-Skłodowska. Sect. A. **9** (1955), 5–28 (1957). (Polish and Russian summaries)

Let Γ be a set of n-square matrices, which contains with each matrix A its transpose A'. The author associates with each A in Γ a generalized canonical matrix $K(A)$ with the following properties: (i) $K(A)$ is similar to A and is in Γ; (ii) $K(A)=K(B)$ when A and B are similar. Any matrix $C(A)$ for which $C^{-1}(A)\cdot A\cdot C(A)=K(A)$ is called a proper matrix of A (by analogy with proper vectors and proper values). For each pair of proper matrices $C(A')$ and $C(A)$, $C'(A)\cdot C(A')$ is a proper matrix of $K'(A)$, and, con-

versely, for each pair $C(A)$, $C(K'(A))$, there exists $C(A')$ such that $C'(A)\cdot C(A')=C(K'(A))$. The author is concerned with the form of $C(K'(A))$ and its implications. For example, when A has distinct eigenvalues, and $K(A)$ is the Jordan canonical form of A, $C(K'(A))$ is diagonal. He generalizes this result, showing that for arbitrary A, when $K(A)$ is either the complex or real Jordan canonical form, $C(K'(A)$ must have a certain form; and conversely, matrices of this form are proper matrices of $K'(A)$.

B. N. Moyls (Vancouver, B.C.).

Whittle, P. Some combinatorial results for matrix powers. Quart. J. Math. Oxford Ser. (2) **27** (1956), 316–320.

Let $A=(a_{jk})$ be a real $p\times p$ matrix such that $a_{jj}>\sum_{k\neq j}|a_{jk}|$. Define

$$F=F(z_1, z_2, \cdots, z_p)=\prod_{j=1}^{p}\sum_{k=1}^{p}\left\{a_{jk}\frac{z_k}{z_j}\right\}^{-1}.$$

Then, if the z's are restricted to lie in certain regions D, F is an analytic function of each of the z's and may be expanded (in D) in an absolutely convergent multiple Laurent series. The author proves that the constant term in the expansion is $(\det A)^{-1}$. This theorem is then applied to a combinatorial problem on matrix powers, involving the determination of the number of ways n_{11} quantities b_{11}, n_{12} quantities b_{12}, $\cdots$, n_{pp} quantities b_{pp} can be ordered so that the first subscript of any b is the same as the second subscript of the b preceding it, while the first and last subscripts of the whole sequence are j and k respectively. Applications to the asymptotic evaluation of certain product sums associated with the elements of the nth power of a matrix are given. *M. Newman.*

Davis, Chandler. All convex invariant functions of hermitian matrices. Arch. Math. **8** (1957), 276–278.

Let H_n denote the real linear space of all hermitian matrices of order n, and let V be a partially ordered real linear space. Let f be a function defined on H_n, taking its values in V such that $f(U^{-1}AU)=f(A)$, for any $A\in H_n$ and for any unitary matrix U of order n. The following result is proved: If the convexity condition $f(tA+(1-t)B)\leqq tf(A)+(1-t)f(B)$, $0\leqq t\leqq 1$, holds for any diagonal matrices A, $B\in H_n$, then it also holds for any arbitrary A, $B\in H_n$. *Ky Fan* (Oak Ridge, Tenn.).

Mirsky, L. Inequalities for normal and Hermitian matrices. Duke Math. J. **24** (1957), 591–599.

This paper is a continuation of an earlier one [Mathematika **3** (1956), 127–130; MR **18**, 460], and deals with the spread $s(A)=\mathrm{Max}_{i,j}|\omega_i-\omega_j|$ of a Hermitian or normal matrix A of order n, where $\{\omega_i\}$ are the eigenvalues of A. The following results are obtained. (1) If A is Hermitian, then $s(A)=2\,\mathrm{Max}\,|(Au, v)|$, when $\{u, v\}$ varies over all pairs of orthonormal vectors. (2) If A is normal and if $t_1, \cdots, t_n$ are n real numbers, then

$$\max_{\pi}\left|\sum_{i=1}^{n}t_{\pi(i)}\omega_i\right|=\max_{\{u_i\}}\left|\sum_{i=1}^{n}t_i(Au_i, u_i)\right|,$$

where π varies over all permutations of the indices $\{1, 2, \cdots, n\}$, and $\{u_i\}$ varies over all systems of n orthonormal vectors. In particular, this implies $s(A)=\mathrm{Max}\,|(Au, u)-(Av, v)|$, where $\{u, v\}$ varies over all pairs of orthonormal vectors. (3) If A is normal, $s(A)\geqq 3^{\frac{1}{2}}\,\mathrm{Max}\,|(Au, v)|$, where $\{u, v\}$ varies over all pairs of orthonormal vectors. The constant factor $3^{\frac{1}{2}}$ is best possible when $n\geqq 3$. Also, a number of lower bounds for $s(A)$ in terms of the elements of A are given. *Ky Fan.*

Jaeckel, K. Nebensymmetrische Matrizen. Z. Angew. Math. Mech. **37** (1957), 400–401.

The author surveys the properties of square matrices which are symmetric with respect to the secondary or sinister diagonal. It is proved that if $A=P\bar{A}'P$ where $P=\bar{P}'=P^{-1}$ and if AP is positive definite, then A has real characteristic roots and linear elementary divisors. When P is the nth order matrix whose (i, j)th element is $\delta_{i,n+1-j}$, this result is applicable to sinister-hermitian (nebenhermitesche) matrices. If, further, A is real, it is applicable to sinister-symmetric (nebensymmetrische) matrices.

D. E. Rutherford (St. Andrews).

Ishaq, Mohammad. Sur les bornes des valeurs caractéristiques de certains produits des matrices et des matrices définies positives. C. R. Acad. Sci. Paris **245** (1957), 480–483.

The author states without a proof some bounds for the characteristic roots of the product of given matrices. But these theorems are immediate corollaries of the corresponding known results for the characteristic roots of a single matrix which are mentioned in the paper. In the definition of α_i' the letter k must be replaced by i and in the definition of θ_j' the letter p must be replaced by k. In the same way, even better results could be obtained by using theorems of A. Brauer [Duke Math. J. **14** (1947), 21–26; **15** (1948), 871–877; **19** (1952), 75–91, 553–562; J. Reine Angew. Math. **192** (1953), 113–116; Proc. Nat. Acad. Sci. U.S.A. **41** (1955), 162–164; MR **8**, 559; **10**, 231; **13**, 813; **14**, 836; **15**, 496; **17**, 1044] and A. Ostrowski [Compositio Math. **9** (1951), 209–226; MR **13**, 524]. Moreover, the following result is given without a proof. Let A and B be positive definite matrices of order n and V their product. Denote the characteristic roots of A by $\alpha_1, \alpha_2, \cdots, \alpha_n$ and those of V by $\mu_1, \mu_2, \cdots, \mu_n$. Then

$$\mu_1\mu_2\cdots\mu_n\leqq\alpha_1\alpha_2\cdots\alpha_n\left\{\frac{\mathrm{tr}\,B}{n(n-1)}\right\}^{n(n-1)}$$

where tr B is the trace of B. It is easy to see that there must be a misprint in this formula. Taking for B the unit matrix, we have $\mu_1\mu_2\cdots\mu_n=\alpha_1\alpha_2\cdots\alpha_n$ and tr $B=n$.

A. Brauer (Chapel Hill, N.C.).

Lotkin, Mark. Characteristic values of arbitrary matrices. Quart. Appl. Math. **14** (1956), 267–275.

The author describes a numerical method for finding the eigenvalues of an arbitrary n-square complex matrix A. The method consists in reducing A to triangular form by a sequence of unitary transformations of a simple type. The triangular part Δ (excluding the principal diagonal) of lower norm is chosen for annihilation, and a suitable pivot element, say A_{pq}, is chosen in Δ. Let T be the unitary matrix obtained from the unit matrix I by replacing I_{pp} by t, I_{qq} by $\bar{t}$, I_{pq} by $-(1-r^2)^{\frac{1}{2}}$ and I_{qp} by $(1-r^2)^{\frac{1}{2}}$, where $r=|t|$. The complex number t is chosen so that the corresponding triangular part of $T^{-1}AT$ has norm less than that of Δ. The process is repeated until a nearly triangular matrix is obtained, in which the diagonal elements are sufficiently close to the eigenvalues of A.

B. N. Moyls (Vancouver, B.C.).

See also: Fields, Rings: Villamayor. **Topological Vector Spaces:** Koechet. **Numerical Methods:** Madić.

Polynomials

Kulik, Stephen. On the Laguerre method for separating the roots of algebraic equations. Proc. Amer. Math. Soc. 8 (1957), 841–843.

An Nth degree polynomial $f(x)$ with N real zeros has, according to Laguerre, at least one zero between numbers u and v which for some real number x satisfies the equation

$$(u-x)(v-x)(f'^2-ff'')+(u+v-2x)ff'+Nf^2=0$$

where $f'=df/dx$ and $f''=d^2f/dx^2$.

The author generalizes this relation by differentiating n times the partial fraction development for

$$[(u-x)(v-x)f'(x)/f(x)]$$

in terms of the zeros of $f(x)$. His result involves the determinants

$$D_n=\begin{vmatrix} f' & f & 0 & \cdots & 0 \\ f'' & f' & f & \cdots & 0 \\ \cdot & \cdot & \cdot & \cdots & \cdot \\ \dfrac{f^{(n)}}{(n-1)!} & \dfrac{f^{(n-1)}}{(n-1)!} & \dfrac{f^{(n-2)}}{(n-2)!} & \cdots & f' \end{vmatrix}.$$

For an arbitrary real x, he finds as one zero A of $f(x)$

$$A=x-\lim_{n=\infty}(E_{n-1}/E_n)$$

where $E_n=(v-x)D_n+fD_{n-1}$. *M. Marden.*

Partial Order, Lattices

Vredenduin, P. G. J. Lattices. Euclides, Groningen 33 (1957/58) 129–152. (Dutch)

An exposition of a branch of "higher mathematics from an elementary point of view".

★ **Maeda, Fumitomo. Kontinuierliche Geometrien.** Die Grundlehren der mathematischen Wissenschaften in Einzeldarstellungen mit besonderer Berücksichtigung der Anwendungsgebiete, Bd. 95. Springer-Verlag, Berlin-Göttingen-Heidelberg, 1958. x+244 pp. DM 36.00.

This book is a translation of the Japanese edition. [Iwanami Shoten, Tokyo, 1952; MR **15**, 540], except that the author has made a few improvements in the exposition. It continues to be the only printed book on the subject of continuous geometry (the original Lecture Notes of J. von Neumann [Continuous geometry, I, II–III, Inst. Advanced Study, Princeton, 1936, 1937] are not yet in print), and it is a most valuable presentation. Unfortunately, it does not cover the important contributions made during the past six years by several writers including S. Maeda, son of the author.

The translation is the work of S. Crampe, G. Pickert, and R. Schauffler. The last named, who was familiar with Japanese, taught S. Crampe enough of the language so that together they could translate the book. Then G. Pickert worked on the translation, with the cooperation of the author. The resulting translation is of very high quality, and the translators are to be heartily congratulated.

The printing is of the excellent standard usual in the Grundlehren Series. *I. Halperin* (Kingston, Ont.).

Rivkind, Ya. I. Dense sublattices of normed Boolean algebras. Grodnenskiĭ Gos. Ped. Inst. Uč. Zap. **1** (1955), 59–66. (Russian)

The first two sections of this article deal with dense subsets of partially ordered sets and dense sublattices of normed Boolean algebras. In the third section the results are applied to the metric theory of functions.

From the introduction.

See also: Foundations, Theory of Sets, Logic: Halmos.

Fields, Rings

★ **Zariski, Oscar; and Samuel, Pierre. Commutative algebra, Volume I. With the cooperation of I. S. Cohen.** The University Series in Higher Mathematics. D. Van Nostrand Company, Inc., Princeton, New Jersey, 1958. xi+329 pp. $7.00.

This is the first of two volumes on commutative algebra designed to provide the necessary background for the senior author's as yet unwritten colloquium book on algebraic geometry. If the first volume is any sample, they will prove to be useful to students and research workers in abstract algebra whether their primary concern is algebraic geometry or not. We begin with a (necessarily incomplete) summary of the contents of the present volume.

In Chapter I, the authors introduce the fundamental concepts with which they will be concerned in both volumes. These include groups, rings, fields, integral domains and quotient fields, unique factorization and Euclidean domains, polynomial rings in several indeterminates, and vector spaces. (The convention that all systems mentioned are assumed to be commutative unless the contrary is mentioned is adopted early in the book.)

Chapter II is concerned with the theory of fields. Algebraic extensions both finite and infinite are considered with special emphasis on fields of finite characteristic. Transcendental extensions also receive a thorough treatment. The chapter concludes with a discussion of algebraic function fields and derivations.

Chapter III begins with a presentation of classical material on ideals and modules of arbitrary (commutative) rings. Direct sums, both finite and infinite are treated thoroughly. Chain conditions and composition series are discussed. The chapter concludes with tensor products of rings and free joins of integral domains.

Chapter IV is about Noetherian rings. In addition to the Hilbert basis theorem, and the Lasker-Noether decomposition theory, it includes a discussion of rings satisfying the descending chain condition, and of quotient rings à la Chevalley and Uzkov. Applications of the latter are given, and the chapter concludes with a discussion of prime ideals in Noetherian rings, principal ideal rings, and an appendix on modules over Noetherian rings.

Chapter V begins with a thorough discussion of integral dependence and integral closure. It is concerned mainly, however, with Dedekind domains (integral domains in which every ideal is a product of prime ideals). Particular attention is paid to finite algebraic extensions of quotient fields of Dedekind domains, and the Hilbert ramification theory. The chapter terminates with number-theoretic applications.

The book includes an informative, well written introduction, which also tells the reader what he can expect to find in Volume II. In addition, there is an index of notation, and an index of definitions to help the reader.

Before the appearance of the present work, the only

systematic account of commutative algebra was to be found in Krull's "Idealtheorie" [Springer, Berlin, 1935]. The reader of "Commutative algebra" will receive a presentation of much of the research in this area over the last twenty years, a good deal of which was inspired by Krull's classic work. In addition, he will receive a leisurely and thorough exposition of the subject matter, suitable not only for the expert, but for the student as well.

Indeed, the present volume is well suited for a graduate level textbook (although the instructor will have to supply his own exercises). Any student who has mastered a substantial part of a book on the level of Birkhoff and MacLane's "A survey of modern algebra" [Macmillan, New York, 1941; MR 3, 99] should have little difficulty in reading the present volume. Motivation and examples are provided. When it seems appropriate, several proofs of a theorem are given. The authors have almost always resisted the temptation to quote "well-known" facts without reference. (One exception occurs in Chapter V, where the reader is expected to know that the ring of Gaussian integers is a Euclidean domain.)

The book contains a number of novel features of which we mention but two. For one thing, the notion of spanning is treated axiomatically, so that it can be applied to both vector spaces and transcendental extensions of fields. In addition, the material on free joins of integral domains is new for the most part.

The authors have given a thorough discussion of the important aspects of commutative algebra without burying them in the midst of an encyclopedic account of the subject. Hence it is inevitable that some people's pet topics be eliminated. This reviewer regrets that the authors did not choose to include Steinitz's characterization of algebraically closed fields [Algebraische Theorie der Körper, de Gruyter, Berlin-Leipzig, 1930] in their excellent chapter on the theory of fields.

In summary, this book is a valuable addition to Van Nostrand's university series in higher mathematics. It deserves a place on every algebraist's bookshelf.

M. Henriksen (Lafayette, Ind.).

Dubois, D. W. On partly ordered fields. Proc. Amer. Math. Soc. 7 (1956), 918–930.

A field F is partly ordered if there is given a non-empty subset P (called the positive cone) which is closed under addition, multiplication, and division; then $x>y$ means $x-y\in P$. If $x+1/n>0$ for all $n=1, 2, \cdots$, implies $x\geqq 0$, then F is said to satisfy condition (AC). When this is the case, P is the intersection of positive cones P_α belonging to simple orderings, and F is isomorphic to the field of quotients of a ring B of real-valued function on a compact space whose points are equivalence classes of the P_α's. Conversely, suppose B is a dense subring of the ring $C(T)$ of continuous real-valued functions on a compact T, such that (α) no non-zero element of B vanishes on an open interval and (β) the equation $bx=a$ has a solution $x\in B$ whenever the set $\{a(t)/b(t); t\in T, b(t)=0\}$ is bounded. Then the field of quotients $\bar{B}$ of B has an (AC) partial order in which B is the ring of bounded elements. In addition to this structure theorem, the paper contains discussions of various basic questions on partly ordered fields, in particular concerning the problem of extending a partial ordering to an extension field.

J. Tate.

Whaples, G. The generality of local class field theory (Generalized local class field theory. V.) Proc. Amer. Math. Soc. 8 (1957), 137–140.

[Parts I-IV: Duke Math. J. **19** (1952), 505–517; **21** (1954), 247–255, 575–581, 583–586; MR **14**, 140; **17**, 464, 465.] A quasi Galois field (qGf) is a perfect field which, in a given algebraic closure, has exactly one (necessarily cyclic) extension of degree n for every natural number n. Since these are the residue fields for which generalized local class field theory holds, it is of interest to show that qGf's are abundant. To do this the author first observes that if k is any field, k^c its algebraic closure, and σ an automorphism of k^c over k of generalized period 0, then the fixed field of σ is a qGf. A method for obtaining such automorphisms is given by the lemma: If for every n a Galois extension K/k contains a cyclic subfield of degree n over k, then K has an automorphism of generalized period 0 over k. Using this lemma, together with the class field theory of function fields in one variable over finite fields, the author proves: Every absolutely algebraic field of characteristic $p>0$ is the absolutely algebraic subfield of some qGf of transcendence degree 1. He also gives a more explicit construction of certain qGf's by means of formal power series over certain special absolutely algebraic fields of characteristic $p>0$ with certain groups of rational numbers as exponents. *J. Tate.*

MacKenzie, R. E.; and Whaples, G. Artin-Schreier equations in characteristic zero. Amer. J. Math. **78** (1956), 473–485.

Let k be a field maximally complete under a valuation with residue field $\bar{k}$ of characteristic $p>0$. Let K/k be a cyclic extension of degree p with generating automorphism σ such that $|\sigma B|=|B|$ for all $B\in K$. Then $|(\sigma-1)B/B|$ assumes a maximum value, m, for $B\in K^*$, and $m^{p-1}\geqq|p|$. Equality holds here if and only if K/k is a Kummer extension. Otherwise, $m\notin|k|$ and K is the splitting field of an irreducible polynomial of the form $f(x)=x^p-x-\lambda$ with $\lambda\in k$, $|\lambda|=m^{-p}$. Conversely, any polynomial of the form $f(x)$ with $|p^p\lambda^{p-1}|<1$ either splits completely or is irreducible with a cyclic splitting field K of degree p, called an Artin-Schreier extension. The nature of the ramification in such an extension is easily described in terms of λ. If $\bar{k}$ is perfect, then every cyclic subfield of a composite of Artin-Schreier extensions is Artin-Schreier. An explicit reciprocity law of local class field theory for Artin-Schreier extensions is given. An appendix contains several examples which show that some of the authors' hypotheses are necessary. *J. Tate.*

Gentile, Enzo R. Homomorphismes et fermeture algébrique des modules à coefficients dans un anneau associatif. Univ. Nac. La Plata. Publ. Fac. Ci. Fisicomat. Serie Segunda. Rev. **5** (1956), 191–200 (1957).

This paper extends results of the reviewer [Ann. of Math. (2) **40** (1939), 360–369], Ikeda and Nakayama [Proc. Amer. Math. Soc. **5** (1954), 15–19; MR **15**, 677], and Villamayor [Rev. Mat. Cuyana **1** (1955), 1–40; MR **18**, 375] dealing with a closure in a ring, the closure being defined in terms of annihilators.

Let A be an associative ring with a unit, and designate A regarded as a left (right) A-module as A' (A''). Let A'^α (A''^α) be the direct sum of α modules isomorphic to A' (A''). If $a=[a_1, a_2, \cdots]\in A'^\alpha$ and $b=[b_1, b_2, \cdots]\in A''^\alpha$, let the inner product $[a, b]$ be $\sum_i a_i b_i$ summed over the

finite number of non-zero summands. For a subset H of A'^{α}, its right annihilator $H^r \in A''^{\alpha}$ is defined by $H^r = \{b;\ b \in A''^{\alpha},\ [a, b]=0 \text{ for all } a \in H\}$. Similarly define the left annihilator $H^l \in A'^{\alpha}$ of a subset $H \in A''^{\alpha}$. For $H \in A'^{\alpha}$, the closure $\bar{H}$ is defined as $(H^r)^l$.

Among properties studied here are the following: $(a)_r$ Every homomorphism of a right ideal $I \subseteq A$ can be obtained by multiplying I on the left by an element of A; $(b)_r$ for right ideals I_1, I_2 in A we have $(I_1 \cap I_2)^l = I_1^l \cup I_2^l$; $(c)_r$ every right ideal in A is closed. It is shown that $(a)_r$ implies that A is injective and that every homomorphism of a module in A''^{n} (n finite) can be obtained as an inner product with an element of A'^{n}. Also $(a)_r$ and $(c)_r$ imply that every submodule of A''^{n} (n finite) is closed. Further applications are made to $m \times n$ matrices over A.

Marshall Hall, Jr. (Columbus, Ohio).

Villamayor, Orlando E. Sur une représentation matricielle de l'anneau d'endomorphismes d'un module quelconque. Univ. Nac. La Plata. Publ. Fac. Ci. Fisicomat. Serie segunda. Rev. **5** (1956), 185–190 (1957).

Let M be an arbitrary right module with a set of β generators over an associative ring A with identity. Then the ring L of A-endomorphisms of M is isomorphic to T'/I, where T' is a subring of the ring of all $\beta \times \beta$ row-finite matrices with elements in A^* (A^* is a ring anti-isomorphic to A). The proof is accomplished by first considering representations for the endomorphisms of a cyclic module, and a direct sum of cyclic modules. The arbitrary module M is then considered as the homomorphic image of a module M', which is the direct sum of cyclic modules. The endomorphisms of M are shown to be identifiable with a subring of the ring of endomorphisms of M'.

K. G. Wolfson (New Brunswick, N.J.).

Villamayor, Orlando. La théorie de Galois pour les anneaux associatifs. Univ. Nac. La Plata. Publ. Fac. Ci. Fisicomat. Serie Segunda. Rev. **5** (1956), 173–184 (1957).

The author develops a "Galois theory" for "general" rings with unit along lines similar to Nakayama's work [J. Math. Soc. Japan **1** (1949), 203–216; MR **12**, 237]: the theory is purely an "outer" one, and it is assumed that the ring has a "normal basis" with respect to each subring consisting of the invariants of an arbitrary subgroup of the Galois group. Under these conditions one can establish the usual correspondence between subgroups of the Galois group and "Galoisian subrings" conveniently defined, at least for an indecomposable ring. There are numerous misprints.

J. Dieudonné (Paris).

Rees, D. A note on form rings and ideals. Mathematika **4** (1957), 51–60.

Q is a fixed local ring with maximal ideal $\mathfrak{m}$; $\mathfrak{q} = (a_1, \cdots, a_n)$ is a fixed $\mathfrak{m}$-primary ideal of Q; $X_1, \cdots, X_n, t$ are independent indeterminates over Q; $R = Q[ta_1, \cdots, ta_n, t^{-1}]$; $\mathfrak{b}^* = \mathfrak{b}Q[t, t^{-1}) \cap R$, where $\mathfrak{b}$ is any ideal of Q; the leading forms of all $F(X) \in Q[X]$ with $F(a) \in \mathfrak{b}$ generate an ideal $I(\mathfrak{b})$ in $Q[X]$; the form ring of Q is defined to be the quotient ring $F = Q[X]/I(0)$, and $I(\mathfrak{b})/I(0)$, considered as an ideal of F, is termed the form ideal of $\mathfrak{b}$ and is denoted by $\bar{\mathfrak{b}}$.

Relations between the ideals $\mathfrak{b}$ and $\mathfrak{b}^*$ are studied, and it is shown that F is isomorphic to $R/t^{-1}R$. The following main result is based on a theorem due to Northcott [Proc. Glasgow Math. Assoc. **2** (1956), 159–169; MR **17**, 938]. Let $\mathfrak{p}_1, \cdots, \mathfrak{p}_r$ be the minimal prime ideals of $\mathfrak{b}$. Let μ_i be the length of the isolated primary component of $\mathfrak{b}$ corresponding to $\mathfrak{p}_i$. Let $\bar{\mathfrak{P}}$ be a minimal prime ideal of $\bar{\mathfrak{b}}$ and let λ, λ_i be the lengths of the isolated components of $\bar{\mathfrak{b}}$ and $\bar{\mathfrak{p}}_i$ resp. which correspond to $\bar{\mathfrak{P}}$, where λ_i is 0 if $\bar{\mathfrak{P}}$ is not a minimal prime ideal of $\bar{\mathfrak{p}}_i$. Then $\lambda = \sum \mu_i \lambda_i$. A geometrical interpretation is given in terms of multiplicities of components of the tangent cone at a point of a variety.

The main result is also used for a new proof of $e(\mathfrak{q}) = \sum \mu_i e((\mathfrak{q}+\mathfrak{p}_i)/\mathfrak{p}_i)$, which is one of the main ingredients of Lech's paper on the associativity formula for multiplicities [Ark. Mat. **3** (1957), 301–314; MR **19**, 11].

F. J. Terpstra (Pretoria).

Yoshida, Michio. A property of polynomial extensions of rings. Proc. Amer. Math. Soc. **8** (1957), 987–989.

Let $A[x] = A[x_1, \cdots, x_n]$ be the ring of polynomials with coefficients in a commutative ring A. The author discusses the relation between the ideals $\mathfrak{a}$ of A and the ideals $\mathfrak{a}A[x]$ of $A[x]$, and proves that: if $\mathfrak{q}$ is primary then $\mathfrak{q}A[x]$ is also primary and both have the same length. This, he shows, can be utilised in some problems of local rings where multiplicities of primary ideals are concerned, to obtain a situation where the residue field of the local ring is infinite, which may bring some simplifications in proof. An example is given.

S. A. Amitsur.

Andrunakievitch, V. A. Contribution to the theory of radicals in associative rings. Dokl. Akad. Nauk SSSR (N.S.) **113** (1957), 487–490. (Russian)

Kuroš [Mat. Sb. N.S. **33**(**75**) (1953), 13–26; MR **15**, 194] and Amitsur [Amer. J. Math. **76** (1954), 100–125; MR **15**, 499] have approached the concept of a radical axiomatically. The author follows Kuroš in his definitions and studies a radical R subject to the condition (I): Every ideal of an R-radical ring is an R-radical ring. Given a radical R, a radical R' is called complementary to R if it is the greatest among all radicals having in any ring a void intersection with R. In case R satisfies (I), the existence of R' is assured and its properties are enunciated. A radical R is said to be dual provided there exists a radical R' such that R and R' are mutually complementary to one another. A radical R satisfying (I) is called a supernilpotent radical provided in any ring K its R-radical contains all nilpotent ideals of K. A radical R satisfying (I) is called a subidempotent radical provided every ideal of an R-radical ring is idempotent. An associative ring having a certain algebraic property σ is said to be a σ-ring; a class of σ-rings is called a special class of rings provided σ satisfies three conditions: (1) every σ-ring is primitive, (2) every non-null ideal of a σ-ring is a σ-ring, (3) every extension K of a non-null σ-ring A is an extension of its annihilator A^* in K by means of a certain σ-ring. Every special class of rings determines an upper radical known as the special radical. Given a supernilpotent (or a subidempotent) radical R, then R' and $R'' = (R')'$ exist and are mutually complementary; moreover, R' is a dual subidempotent radical and R'' is a special radical, the smallest dual radical containing R. If φ is an arbitrary algebraic property, let M_φ be the class of all φ-rings which are subdirectly irreducible with the additional property that the intersection of all non-null ideals of the ring is idempotent. Then M_φ is a special class of rings. If $M_{\tilde{\varphi}}$ is the class of all other subdirectly irreducible rings, then M_φ and $M_{\tilde{\varphi}}$ determine respectively upper radicals R_φ and $R_{\tilde{\varphi}}$ which are mutually complementary; R_φ is a dual special radical, R_φ is a dual subidempotent radical. Furthermore, all

dual supernilpotent and subidempotent radicals are obtainable in this manner. If R is a given supernilpotent radical and K is a strongly R-semisimple ring and if all the ideals of K whose factor-rings are non-null subdirectly irreducible rings have annihilators different from zero, then K is the discrete direct sum of simple rings belonging to a suitable special class of rings. Conversely every discrete direct sum of simple rings belonging to a given special class M of rings is a strongly R-semisimple ring, where R is the special radical determined by the class M. *R. A. Good.*

See also: Homological Algebra: Northcott.

Algebras

Patterson, E. M. Generators of linear algebras. Proc. London Math. Soc. (3) **7** (1957), 467–480.

M. S. Knebelman [Ann. of Math. (2) **36** (1935), 46–56] has defined the genus of an n-dimensional linear algebra A to be the difference between its dimension and the least number of linearly independent elements whose products generate A. The author has previously studied algebras of genus zero [J. London Math. Soc. **31** (1956), 326–331; MR **18**, 109] and algebras of genus 1 with unit element [ibid. **32** (1957), 88–94; MR **18**, 638]. In the present paper he proves the following: (I) Let A be a linear algebra of genus r and dimension n over any field F. If $n \geqq 3r+3$ there exist $r+s$ elements $a_1, a_2, \cdots, a_{r+s}$ in A, with $s \leqq 2r$, such that multiplication in A is given by (α) $xy = x\phi(y)+y\psi(x)+\sum_{h=1}^{r+s}\theta^h(x, y)a_h$, where ϕ and ψ are linear functions, θ^h are bilinear functions and $a_1, \cdots, a_r$ are generated by $a_{r+1}, \cdots, a_{r+s}$. If $F \neq GF(2)$, the theorem is also true for $n=3r+2$. (II) Let A be an algebra of genus 1 and dimension $n(\geqq 6)$. Then there exist linear functions ϕ and ψ, a bilinear function θ, not identically zero, and a fixed non-zero $c \in A$ such that multiplication in A is given by (β) $xy = x\phi(y)+y\psi(x)+c\theta(x, y)$. Conversely, if A is an algebra in which multiplication is given by (β), with ϕ, ψ, θ and c as above, then A has genus 1. The paper also includes a complete classification, according to multiplication structure and by means of formulas such as (α) and (β), of all associative algebras of genus 1 and dimension $n \geqq 6$. Other pertinent remarks are included.

R. L. San Soucie (Buffalo, N.Y.).

See also: Fields, Rings: Zariski and Samuel. **Lie Groups and Algebras:** Gurevič.

Groups and Generalizations

Ree, Rimhak. Commutator groups of free products of torsion-free abelian groups. Ann. of Math. (2) **66** (1957), 380–394.

Let F be a free product of an arbitrary set of torsion-free abelian groups, F' the first, F'' the second derived group of F. As main results, the paper contains the following theorems. (1) The lower central series $G=G_1, G_2, \cdots$ of the group $G=F/F''$ converges to the unit group and the groups G_i/G_{i+1} ($i=1, 2, \cdots$) are torsion-free. (2) If F is the free product of a finite number of free abelian groups $U_1, \cdots, U_m$ of finite ranks $r_1, \cdots, r_m$ respectively, then the rank N_i of G_i/G_{i+1} is given by the formula

$$N_i=(m-1)\binom{i+r-1}{i}-\sum_{k=1}^{m}\binom{i+r_k-1}{i},$$

where $r=r_1+\cdots+r_m$. (3) If the elements $g_1, \cdots, g_s$ in F represent distinct cosets mod F', and for $f \in F'$ and for the integers $m_1, \cdots, m_s$ the relation $\prod_{i=1}^{s} g_i f^{m_i} g_i^{-1} \in F''$ holds, then either $f \in F''$ or $m_1=\cdots=m_r=0$. — The results of the paper are based on the investigation of the structure of the R-module M constructed as follows: R is the group ring of F/F' over the ring of rational integers, $M=F'/F''$ (which is free abelian) and the operation rm ($r \in R$, $m \in M$) is defined in the natural way by the aid of the inner automorphisms of F. — Theorems (1), (2) generalize results due to K.-T. Chen [Ann. of Math. (2) **54** (1951), 147–162; MR **13**, 105]. *A. Kertész.*

Zappa, Guido. Sugli automorfismi privi di coincidenze nei gruppi finiti. Boll. Un. Mat. Ital. (3) **12** (1957), 154–163.

It is proved that there are nilpotent finite groups, of arbitrarily high class, possessing automorphisms that leave only unity fixed, and that a finite supersolvable group possessing such an automorphism of prime order is nilpotent. *L. J. Paige* (Los Angeles, Calif.).

Minc, H. Index polynomials and bifurcating root-trees. Proc. Roy. Soc. Edinburgh, Sect. A. **64** (1957), 319–341.

The logarithmetic of a nonassociative groupoid has been defined by Etherington [see, e.g. same Proc. **64** (1954–55), 150–160; MR **17**, 825]. If $\mathfrak{A}$ is a free cyclic multiplicative groupoid, its logarithmetic $\mathfrak{L}$ is an algebra with additive groupoid isomorphic to $\mathfrak{A}$ and in which multiplication is associative and right distributive with respect to addition. The elements of $\mathfrak{L}$ are represented diagramatically by "trees". Addition and multiplication of trees is defined to give a system isomorphic to $\mathfrak{L}$. The trees in turn are represented in various ways by polynomials in the domain $\mathfrak{M}[\lambda, \mu]$. Here $\mathfrak{M}$ is the ring of integers and λ, μ are noncommuting indeterminates. In each such representation discussed, addition and multiplication of trees are shown to correspond to certain binary operations on polynomials in $M[\lambda, \mu]$. An index, or tree, P is prime if $P \neq 1$ and $P=QR$ implies $Q=1$ or $R=1$. Uniqueness of factorization into prime trees is proved by considering one of these polynomial representations. It is shown that the set L of all trees is a distributive lattice and also that a metric can be defined on L. *D. C. Murdoch* (Vancouver, B.C.).

Gluskin, L. M. Elementary generalized groups. Mat. Sb. N.S. **41(83)** (1957), 23–36. (Russian)

Homological Algebra

Northcott, D. G. A note on the global dimension of polynomial rings. Proc. Cambridge Philos. Soc. **53** (1957), 796–799.

In this note, another proof is given of the fact that if K is a field and $X_1, \cdots, X_n$ are indeterminants, then the global dimension (in the homological sense) of the polynomial ring $K[X_1, \cdots, X_n]$ is n. The proof given here employs more standard commutative ring theory and less homological algebra than most that have been given. [For a more homological proof see, e.g., Cartan and Eilenberg, Homological algebra, Princeton, 1956, Ch. IX, Th. 7.11; MR **17**, 1040.] *M. Auslander.*

See also: Theory of Algebraic Numbers: Nakayama. **Algebraic Topology:** Palermo.

THEORY OF NUMBERS

General Theory of Numbers

Jacobsthal, Ernst. Über eine zahlentheoretische Summe. Norske Vid. Selsk. Forh., Trondheim **30** (1957), 35–41.

If $[x]$ is the greatest integer $\leqq x$, a, b are integers and m, r positive integers, the author proves the inequality

$$\sum_{h=0}^{r-1}\left(\left[\frac{a+b+h}{m}\right]-\left[\frac{a+h}{m}\right]-\left[\frac{b+h}{m}\right]+\left[\frac{h}{m}\right]\right)\geqq 0.$$

H. D. Kloosterman (Leiden).

Jacobsthal, Ernst. Über die grösste ganze Zahl. I. Norske Vid. Selsk. Forh., Trondheim **30** (1957), 1–5.

For integers n, N and real x, y let

$$f(n)=[n(x+y)]-[nx]-[ny],\quad F(N)=\sum_{n=1}^{N}f(n)$$

($[z]$ is as usual the greatest integer $\leqq z$). Then the limit

$$\lim_{N\to\infty}N^{-1}F(N)=A(x,y)$$

exists. The author proves: (a) $A(x,y)=0$ if and only if at least one of the two real numbers x, y is an integer; (b) if neither x nor y is an integer, then $A(x,y)\geqq\frac{1}{6}$ and there exist real values of x and y for which $A(x,y)=\frac{1}{6}$; (c) $A(x,y)=1$ if and only if x and y are irrational numbers such that $x+y$ is an integer. *H. D. Kloosterman.*

Jacobsthal, Ernst. Über die grösste ganze Zahl. II. Norske Vid. Selsk. Forh., Trondheim **30** (1957), 6–13.

If as usual $[z]$ denotes the greatest integer $\leqq z$, the author determines the inverse A_n^{-1} of the matrix A_n with elements

$$a_{rs}=\left[\frac{r}{s}\right]\quad(r,s=1,2,\cdots,n).$$

We have $A_n^{-1}=(b_{\kappa\lambda})$, where

$$b_{\kappa\lambda}=\mu\left(\frac{\kappa}{\lambda}\right)-\mu\left(\frac{\kappa}{\lambda+1}\right)\quad(\kappa,\lambda=1,2,\cdots,n)$$

and $\mu(x)$ is the Moebius function which for non-integral x is to be taken as 0. For the infinite matrix $A=(a_{rs})$ $(r,s=1,2,\cdots)$ he proves: (a) If m is a given positive integer, exactly $\frac{1}{2}(m-1)(m+2)$ rows of A contain no element $=m$; (b) exactly $m(m+1)$ rows of A contain m exactly p times $(p=1,2,3,\cdots)$. He also proves some results about the indices of these rows.

H. D. Kloosterman (Leiden).

Hornfeck, Bernhard; und Wirsing, Eduard. Über die Häufigkeit vollkommener Zahlen. Math. Ann. **133** (1957), 431–438.

Let $\sigma(n)=\sum_{d|n}d$. For any set A of positive integers, $A(x)$ denotes the number of elements $\leqq x$ of A. For any rational $\kappa>1$, let V_κ be the set of all n with $\sigma(n)=\kappa n$. Thus V_1 is the set of perfect numbers. Let S be the set of all n with $n|\sigma(n)$. The author's main result $V_\kappa(x)=O(\exp(c\cdot\ln x\cdot\ln\ln x/\ln\ln x))$ readily yields the same estimate for $S(x)$. It implies in particular $V_\kappa(x)=O(x^\varepsilon)$ and $S(x)=O(x^\varepsilon)$ for every $\varepsilon>0$. The best earlier estimates seem to be due to Kanold [Math. Ann. **132** (1957), 442–450; MR **18**, 873]: $V_1(x)=O(x^{1/4}\ln x/\ln\ln x)$ and $S(x)=O(x^{\frac{1}{2}}\cdot\ln\ln x)$. *P. Scherk* (Saskatoon, Sask.).

Carlitz, L. A special quartic congruence. Math. Scand. **4** (1956), 243–246.

The author recently [Amer. Math. Monthly **63** (1956), 569–571; MR **18**, 379] studied the reducibility of the quartic polynomial x^4+ax^2+b modulo a prime number p. In this paper, he solves the corresponding problem for the reciprocal quartic polynomial

$$x^4+ax^3+bx^2+ax+1.$$

The result for the special case $a=b=1$ has been known for a long time [see, e.g., H. J. S. Smith, Collected mathematical papers, v. 1, Oxford, 1894, p. 103]. *A. Brauer.*

Leech, John. Some solutions of Diophantine equations. Proc. Cambridge Philos. Soc. **53** (1957), 778–780.

The author considers the Diophantine equation

$$f(x)+f(y)=f(z)+f(t),$$

where $f(x)$ is a power of x or a binomial coefficient $\binom{x}{n}$, n fixed. A search for solutions was carried out on EDSAC $1\frac{1}{2}$ at the University Mathematical Laboratory, Cambridge. Thus for example the author finds that the equation

$$\binom{x}{5}+\binom{y}{5}=\binom{z}{5}+\binom{t}{5}$$

has three solutions in integers less than 500, given by

x	y	z	t
9	9	10	
118	117	133	78
197	160	209	53.

Other cases treated were

$$f(x)=x^3,\ x^4,\ \binom{x}{3},\ \binom{x}{4}.$$

A consequence of the author's work is that the solution

$$158^4+59^4=134^4+133^4,$$

due to Euler, is in fact the one with the smallest integers. *M. Newman* (Washington, D.C.).

Chalk, John. H. H. Quelques équations de Pell généralisées. C. R. Acad. Sci. Paris **244** (1957), 985–988.

The author extends the results obtained in an earlier paper [Math. Ann. **132** (1956), 263–276; MR **18**, 718] for the rational complex field $k(i)$ to the field $k(i\sqrt{P})$, where P is a square-free positive integer. By means of similar arguments and a corresponding identity of G. Humbert [C. R. Acad. Sci. Paris **171** (1920), 287–293, 377–382], he considers solutions of the equation $uu'-Dvv'=1$ in algebraic integers u, v and their conjugates u', v', where D is a fixed integer. He shows that there exists a solution for which $\sqrt{(uu')}$ is bounded above by an upper bound depending on D and P which is similar to, but more complicated than, the corresponding upper bound when $P=1$. When $P\equiv 3 \pmod 4$ this gives an upper bound for $\sqrt{(x^2+Py^2)}$, where x, y, z and w are rational integers for which

$$x^2+Pz^2-D(z^2+w^2)=1.$$

By transforming this equation upper bounds for small solutions of certain other equations are obtained. *R. A. Rankin* (Glasgow).

Dijkstra, E. W. A method to investigate primality. Math. Tables Aids Comput. **11** (1957), 195–196.

This is a generalization of a method of G. G. Alway [Math. Tables Aids Comput. **6** (1952), 59–60] for finding the smallest odd prime factor of a number. The method is based on recurrence relations between successive remainders which do not involve divisions, thus speeding up the computation. *B. A. Galler* (Ann Arbor, Mich.).

Watson, G. L. The equivalence of quadratic forms. Canad. J. Math. **9** (1957), 526–548.

This is a sequel to a previous paper [Mathematika **2** (1955), 32–38; MR **17**, 128], which dealt with ternary indefinite forms and in which some of the basic ideas and notations are introduced. The author here also uses Brandt's definition of the determinant d of the form; he lets Γ denote the set of all square-free integers under the operation $v_1 \cdot v_2 = v_1 v_2 (v_1, v_2)^{-2}$ and Γ_d the subgroup of Γ where $(v, d)=1$. Then groups $\Gamma(f)$ and $\Gamma^+(f)$ are defined for which the following theorem holds: Theorem 1: Let f be a non-degenerate quadratic form, with integral coefficients, in at least three variables. Then (i) the number of classes in the genus of f is not less than the order of the factor group $\Gamma_{d(f)}/\Gamma(f)$ or $\Gamma_{d(f)}/\Gamma^+(f)$, when improper equivalence is or is not, respectively, admitted; (ii) $\Gamma(f)=\Gamma^+(f)$ is a necessary condition for f to be improperly equivalent to itself; (iii) if f is indefinite, then there is equality in (i) and the necessary condition in (ii) is also sufficient.

The author also proves Theorem 4: Suppose that f is indefinite, k (the number of variables) not less than 3; let d_1 be the greatest integer whose $(k(k-1)/2)$th power divides d. Suppose also that $d_1=1$, 2, 4, p or $2p$; $p \equiv -1 \pmod 4$. Then the class-number of f, in the strict sense (determinant 1), is 1. *B. W. Jones.*

Kneser, Martin. Klassenzahlen definiter quadratischer Formen. Arch. Math. **8** (1957), 241–250.

The author considers positive definite quadratic forms in n variables of determinant d, over the ring of rational numbers whose denominators are powers of 2. Using extensions of the reduction theory and other methods, he determines the class number for all $d \leqq 3$ with $n+d \leqq 17$. In particular, the class number is 1 for $n \leqq 6$ and $d=1$ or 2; for $d=1$ and $n=7$; for $d=3$ and $n=1$. *B. W. Jones.*

McCarthy, Paul J. On indefinite ternary genera of one class. Math. Z. **68** (1957), 290–295.

The author gives a set of theorems stating sufficient conditions that an indefinite ternary form be in a genus of one class. Here J is the g.c.d. of the two-rowed minor determinants of the form and K is determined by the fact that J^2K is equal to the determinant. Two of the results are: A ternary indefinite quadratic form is in a genus of one class if 2 is the g.c.d. of J and K and if one of the following hold:

$$J \equiv 8 \pmod{16} \text{ and } K \equiv 2 \pmod 4,$$
$$J \equiv 2 \pmod 4 \text{ and } K \equiv 4 \pmod 8.$$

B. W. Jones (Boulder, Colo.).

Kolberg, Oddmund. Some identities involving the partition function. Math. Scand. **5** (1957), 77–92.

Let $\varphi(x)=\prod_{n=1}^{\infty}(1-x^n)=\sum_{n=-\infty}^{\infty}(-1)^n x^{\frac{1}{2}n(3n+1)}$, so that $\varphi(x)^{-1}$ generates the partition function $p(n)$. Let q be a prime and define, for $0 \leqq s \leqq q-1$,

$$g_s = \sum_{\frac{1}{2}n(3n+1) \equiv s \pmod q} (-1)^n x^{\frac{1}{2}n(3n+1)};$$

$$P_s = \sum_{n=0}^{\infty} p(qn+s) x^{qn+s},$$

so that $P_0+P_1+\cdots+P_{q-1}=\varphi(x)^{-1}$; $g_0+g_1+\cdots+g_{q-1}=\varphi(x)$. By elementary methods of great elegance the author derives a number of identities for the functions P_s principally for $q=2$, 3, 5, 7. These include the Ramanujan identities for partitions modulo 5, 7 and many others, such as

$$(1) \quad \sum_{n=0}^{\infty} \{p(5n+1)x^{5n}+p(5n+2)x^{5n+1}\} = \varphi(x^5)^{-6}\{\varphi(x)^3\varphi(x^{25})^2+5x\varphi(x)^2\varphi(x^{25})^3+10x^2\varphi(x)\varphi(x^{25})^4 +10x^3\varphi(x^{25})^5\};$$

$$(2) \quad \sum_{n=0}^{\infty} p(3n)x^n = \varphi(x)^{-4}\varphi(x^3)\varphi(x^9)^2 \prod_{n=1}^{\infty}(1-x^{9n-5})^2 \times (1-x^{9n-4})^2 - x\varphi(x)^{-3}\varphi(x^9)^2 \prod_{n=1}^{\infty}(1-x^{9n-5})^{-1}(1-x^{9n-4})^{-1}.$$

Formula (1), for example, may certainly be used to obtain information about $p(5n+1)$, $p(5n+2)$ modulo 5 not derivable in any standard way from the theory of the elliptic modular functions (which only yield information about $p(5n+4)$ modulo 5). Unfortunately the new identities for $q=7$ derived by the author do not seem useful for such a purpose, since they all involve products of the P's. *M. Newman* (Washington, D.C.).

Analytic Theory of Numbers

Uchiyama, Saburô. On a multiple exponential sum. Proc. Japan Acad. **32** (1956), 748–749.

Let p be a prime and k a finite field with $q=p^r$ elements. Writing $e(\alpha)=\exp 2\pi i\, t(\alpha)/p$, where $t(\alpha)$ denotes the trace of $\alpha \in k$, the author estimates the sum $S_m(f)=\sum e(f(x_1, \cdots, x_m))$, where f is a polynomial in $k[x_1, \cdots, x_m]$ of degree n less than p and greater than 1, and where $x_1, \cdots, x_m$ run independently over all elements of k. His result $S_m(f)=O(q^{m-\frac{1}{2}})$ follows immediately from a result (for $m=1$) of L. Carlitz and the author [Duke Math. J. **24** (1957); 37–41; MR **18**, 563]. *H. D. Kloosterman.*

Maass, Hans. Zetafunktionen mit Grössencharakteren und Kugelfunktionen. Math. Ann. **134** (1957), 1–32.

In a previous paper [J. Indian Math. Soc. (N.S.) **20** (1956), 117–162; MR **19**, 252] the author discussed the problem whether the series

$$\phi_0(s, S; u, v) = \sum_G u(QG)\, v(S[G]) |S[G]|^{-s-k}$$

define zeta-functions, i.e., meromorphic functions which satisfy a functional equation of Riemann's type. Here $u(X)$ denotes a spherical function of type (m, n) and degree $2nk$, $v(Y)$ an angular character of quadratic forms, S a positive matrix of type $S^{(m)}$ and Q the positive matrix defined by $S=Q'Q$. G runs through a complete system of integral right non-associated matrices of type $G^{(m,n)}$ and rank n. The answer is affirmative if $u(X)$ is harmonic. The problem for an arbitrary spherical function $u(X)$ is related to that involving polynomials $u(X)$ with the invariance property $u(XV)=u(X)$ for orthogonal matrices V. In this case generalized theta-functions related to the problem can be defined. In general however the application of Mellin's transformation to these functions leads not to ϕ_0 but to a different function ϕ. If $u(X)$ satisfies the condition $u(XV)=|V|^{2k}u(X)$ for $|V| \neq 0$, ϕ and ϕ_0 coincide apart from an elementary factor, and a result of the required type is obtained. *E. C. Titchmarsh.*

Mitrović, Dragiša. Sur la fonction ζ de Riemann. C. R. Acad. Sci. Paris **245** (1957), 885–886.

In this note bounds for the derivatives of $\zeta(s)$ and $1/\zeta(s)$ in the half-plane $\sigma>1$ are obtained, and results are stated about the signs of the coefficients in the Laurent series for $\zeta(s)$ in powers of $s-1$. These correct in certain respects the results stated in a previous note [same C. R. **244** (1957), 1602–1604; MR **19**, 393]. *E. C. Titchmarsh.*

Vinogradov, I. M. Trigonometric sums involving values of a polynomial. Izv. Akad. Nauk SSSR. Ser. Mat. **21** (1957), 145–170. (Russian)

Let $f(x)=A_nx^n+\cdots+A_1x$ be a real polynomial of degree n without constant term. Let P and k be positive integers. In the present paper the author improves his previous estimates [same Izv. **20** (1956), 289–302; MR **18**, 381] of trigonometric sums

$$S=\sum \exp\{2\pi ikf(x)\},$$

where x runs through all integers, or else through all primes, from 1 to P. (The case $n=1$ was a key step in his attack on the Goldbach conjecture.) Each theorem exhibits two different estimates, depending on whether the coefficients $A_n, \cdots, A_1$ of f are of the "first class" or not. For instance, Theorem 2 (in which the sum S is taken over primes, and which the author designates as the principal theorem of the paper) puts $A_n, \cdots, A_1$ in the first class when they can be approximated by irreducible fractions $a_n/q_n, \cdots, a_1/q_1$ with the l.c.m. Q of $q_1, \cdots, q_n$ not exceeding $P^{1/n}$ and with each $|A_s-a_s/q_s|\leq P^{1/n}/P^s$. In this case, letting $r=1/(62^2\ln 12n^2)$, and keeping $k\leq P^{2r}$, he proves $S=O(P^{1-r})$. On the other hand, when $A_n, \cdots, A_1$ cannot be so approximated he imposes the weaker restriction $k\leq Q^{1/r}$ and proves $S=O(P^{1+\varepsilon}((k, Q)/Q)^{1/2n})$, for each positive ε. The author also states that a substantial further improvement of the estimate for S is possible when $A_n, \cdots, A_1$ belong to the first class. *H. Mirkil.*

Hooley, C. An asymptotic formula in the theory of numbers. Proc. London Math. Soc. (3) **7** (1957), 396–413.

Let $d(n)$ be the number of divisors of n and $d_3(n)$ be the number of representations of n as the product of three factors. In this paper the author investigates the difficult problem of obtaining asymptotic formulae for the sums

$$A_0(x)=\sum_{n\leq x} d_3(n)d(n+a),\quad B_0(x)=\sum_{n\leq x} d_3(n+a)d(n).$$

His results confirm formulae involving these sums which were conjectured earlier by Titchmarsh [Quart. J. Math. Oxford Ser. **13** (1942), 129–152; MR **4**, 131]. The main theorem is as follows. As $x\to\infty$,

$$A_0(x)=\tfrac{1}{2}A(a)x\log^3x+O\{x(\log x\log\log x)^2\},$$

where $A(m)=\sum_{n=1}^{\infty} c_n(m)\psi(n)/n^2$, $\psi(n)=\sum_{d|n}\phi(d)/d$, and $c_n(m)$ is the Ramanujan sum. This theorem is also valid when $A_0(x)$ is replaced by $B_0(x)$. The analysis depends upon a detailed study of the auxiliary sums

$$C_\alpha(x, k)=\sum_{\substack{n\leq x\\ n\equiv a \pmod k}} n^\alpha d(n),\quad D_\alpha(x, k)=\sum_{\substack{n\leq x\\ n\equiv a \pmod k}} n^\alpha d^2(n),$$

where α is a suitable positive constant. The estimate of the Kloosterman sum due to Weil [Proc. Nat. Acad. Sci. U.S.A. **34** (1948), 204–207; MR **10**, 234] is also required. By employing similar methods the author has derived formulae for the 'conjugate' sum $\sum_{v=1}^{m-1} d_3(m-v)d(v)$ and for the sums $\sum_{n\leq x} d_3(n)r(n+a)$, $\sum_{v=1}^{m-1} d_3(m-v)r(v)$, where $r(n)$ is the number of representations of n as the sum of two integral squares. *A. L. Whiteman.*

Kanold, Hans-Joachim. Über zahlentheoretische Funktionen. II. Math. Ann. **134** (1957), 41–46.

[For Part I see J. Reine Angew. Math. **195** (1955), 180–191; MR **17**, 827.] The author considers functions $f(n)$ defined for positive integers n and whose values $f(n)$ are also positive integers. The set of values of f is denoted by $\mathfrak{F}$. The function is called multiplicative if $(n_1, n_2)=1$ implies $f(n_1, n_2)=f(n_1)f(n_2)$. The following theorem is proven: Let K and $a>1$ be positive integers and let $f(n)$ be multiplicative, having in addition the following properties: a) for all primes $p\leq K$ and $\alpha=1, 2, \cdots$, we have $f(p^\alpha)=p^\alpha$; b) if $q_1<q_2<\cdots$ are the primes $>K$, then $f(q_i)=ia$ $(i=1, 2, \cdots)$; c) for all primes $q>K$ and $\beta=2, 3, 4, \cdots$, we have $f(q^\beta)=aq^\beta$. Then for every $\varepsilon>0$ we have

$$\liminf \frac{f(n)}{n^{1-\varepsilon}}>\liminf \frac{f(n)}{n}\prod_{q|n}\log q>0 \quad (n\to\infty)$$

and also

$$D^*(\mathfrak{F})=a^{-1}$$

($D^*(\mathfrak{F})$ denotes the asymptotic density of $\mathfrak{F}$, i.e., $\lim_{N\to\infty} F(N)/N$, where $\mathfrak{F}(N)$ is the number of elements $\leq N$ in $\mathfrak{F}$). — In addition, the author proves two more theorems on asymptotic densities, which are improvements on theorems in Part I [loc. cit.].

H. D. Kloosterman (Leiden).

Shimura, Goro. La fonction ζ du corps des fonctions modulaires elliptiques. C. R. Acad. Sci. Paris **244** (1957), 2127–2130.

Let M be the field of all elliptic modular functions of level ("Stufe") n. Then M is a field of algebraic functions in one variable over the field C of complex numbers. The author constructs a subfield K of M having the field of rational numbers as its field of constants, which further generates M over C and such that its zeta function can be represented by means of the Riemann ζ-function and an Euler product [as introduced by Hecke, Math. Ann. **114** (1937), 1–28, 316–351]. In addition, the absolute values of the characteristic roots of Hecke's operator T_p for the cusp forms of dimension -2 (differentials of the first kind) do not exceed $2\sqrt{p}$ for almost all prime numbers p.

H. D. Kloosterman (Leiden).

★**van Lint, Jacobus Hendricus. Hecke operators and Euler products.** Drukkerij "Luctor et Emergo", Leiden, 1957. 51 pp. (1 insert)

Wohlfart [Dissertation, Münster, 1955] has given a theory of generalized Hecke operators of not necessarily integral dimension. While the original Hecke operators map the set of integral modular forms of dimension $-r$ and step 1 into itself, the generalized operators $T_K^\Lambda(Q)$ map the set K of forms $\{\Gamma, -r, v\}$ into the set Λ of forms $\{\theta, -r, v^*\}$, where Γ and θ are subgroups of the modular group $\Gamma(1)$, Q is a matrix with rational elements and positive determinant, and

$$v^*(L)=\frac{\sigma(QLQ^{-1}, Q)}{\sigma(Q, L)}v(QLQ^{-1})$$

for any matrix $L\in\Delta$, where $\Delta=\theta\cap\Gamma(1)\cap Q^{-1}\Gamma Q$ and is of finite index in $\Gamma(1)$ and σ stands for a certain root of unity. In general (with one important exception), all Hecke operators defined on the class $\{\Gamma, -r, v\}$ can be built up from operators $T_K^\Lambda(Q)$ defined by $f|T_K^\Lambda(Q)=\sum v^*(V)^{-1}f|QV$, the summation being extended over a complete system V of representatives of a decomposition of θ into left cosets of Δ. If Γ and θ are arbitrary subgroups

of $\Gamma(1)$, in general no non-trivial such operators exist; if they exist, then the determination of Q and, particularly, the proof that v^* has the required value are the main difficulties. Some of the particular cases studied are $\Gamma=\theta=\Gamma(1)$, r integral, $Q=\begin{pmatrix} p & 0 \\ 0 & 1 \end{pmatrix}$ (operators denoted $T(p)$) and $Q=\begin{pmatrix} p^2 & 0 \\ 0 & 1 \end{pmatrix}$ (operators denoted $T(p^2)$); $\Gamma=\theta=\Gamma(1)$, or $\Gamma=\theta=\Gamma_0(2)$, r half odd integer, $Q=\begin{pmatrix} p^2 & 0 \\ 0 & 1 \end{pmatrix}$; $\Gamma=\Gamma(1)$, $\theta=\Gamma_0(p)$, r half-odd integer, $Q=\begin{pmatrix} 1 & 0 \\ 0 & p \end{pmatrix}$, with $p>3$, prime. This theory is used (prove existence of operator, apply it to a function of the first set, compare coefficients of the two members) and permits the derivation in a systematic way of many known results and some new ones.

Applying the operators to even powers $l=2r$ of the (Dedekind) η-function,

$$\eta^l(\tau)=\sum_{0<n\equiv l \pmod{24}} a(n)e^{2\pi i\tau n/24}$$

(observe that $\eta^l\in\{\Gamma(1), -r, v\}$ with integral r; hence, also the classical Hecke operator T_p could be used successfully), the author obtains several theorems due to Newman [Nederl. Akad. Wetensch. Proc. Ser. A. **59** (1956), 204–216; J. London Math. Soc. **30** (1955), 488–493; **31** (1956), 205–208, 350–359; MR **17**, 946, 15; **18**, 194] and some new results. Sample theorems: If $0<l\leqq 24$, l even, then, for $l(p-1)\equiv 0 \pmod{24}$, $\eta^l(\tau)$ is an eigenfunction of $T(p)=(-1)^{r(p-1)/2}pT_p$ for all p; otherwise $\eta^l(\tau)$ is still an eigenfunction of $T(p^2)$ for all $p>3$ and also for $p=3$ if $3|l$. $\eta^{10}(\tau)$ is an eigenfunction for $p\not\equiv 5 \pmod{12}$, while for $p\equiv 5 \pmod{12}$,

$$\eta^{10}(\tau)=p\cdot a(2p)G_4(\tau)\eta^2(\tau)$$
$$=\tfrac{1}{2}\{\eta^{10}(\tau)+G_4(\tau)\eta^2(\tau)/48\}+\tfrac{1}{2}\{\eta^{10}(\tau)-G_4(\tau)\eta^2(\tau)/48\}.$$

The Dirichlet series connected with the two functions in braces have Euler products, too complicated to be quoted. Setting $\prod_{n=1}^{\infty}(1-x^n)^l=\sum_{n=0}^{\infty}p_l(n)x^n$ and $\eta^l(\tau)=\sum_{n=0}^{\infty}a(n)e^{2\pi i\tau n/24}$, one has $p_l(n)=a(24n+l)$. As a corollary of previous results one has, in particular, $p_{10}(n)=0$ if some prime $\equiv 7$ or $\equiv 11 \pmod{12}$ divides $24n+10$ in an odd power, or if all primes $\equiv 5 \pmod{12}$ divide $24n+10$ in even powers. The powers $\eta^l(\tau)$, $l=14$, 16, 18 and 20 are similarly studied. For odd powers, $\eta^l(\tau)\in\{\Gamma(1), -r, v\}$, r is half an odd integer and use of the generalized operators is essential. While, in general, $T(p)$ cannot be defined, $T(p^2)$ exists and its application to $\eta^l(\tau)$ leads to the result: If $0<l<24$ and l is odd, then $\eta^l(\tau)$ is an eigenfunction of $T(p^2)$ for all $p>3$ and also for $p=3$, if $3|l$. Among the consequences for $p_l(n)$, the following may be quoted: $p_{15}(53n^{2k}+15(n^{2k}-1)/24)=0$ (obtained using Newman's result $p_{15}(53)=0$), and $p(5n+4)\equiv 0 \pmod 5$ and $p(7n+5)\equiv 0 \pmod 7$, as was conjectured by Ramanujan and proven by Rademacher [Trans. Amer. Math. Soc. **51** (1942), 609–636; MR **3**, 271]. Application of $T(p^2)$ to $\vartheta^l(\tau)=\sum_{n=0}^{\infty}r_l(n)e^{\pi i\tau n}$ ($r_l(n)$=number of representations of n as sum of l squares), l odd, $l<8$, leads to proofs of the (known) formulae for $r_l(p^2)$ ($l=3$, 5, 7). Finally, a study is made of modular forms of non-integral dimensions that have an Euler product. Among the results obtained are: $\eta(\tau)$ and $\eta^3(\tau)$ are the only integral modular forms $\in\{\Gamma(1), -r, v\}$, for which linear relations $f(p\tau)+\alpha\sum_{b=0}^{p-1}f((\tau+Nb)/p)=0$ hold for all primes p with $(p, N)=1$. The book ends with a tabulation of the integral modular forms $\{\Gamma_\theta, -r, v\}$, with r a half odd integer and Euler product of the form $\prod_{(p,N)=1}(1+\alpha^{-1}(p)p^{-1-2s})^{-1}$, where Γ_θ is generated by $\begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}$ and $\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$.

E. Grosswald (Philadelphia, Pa.).

Plünnecke, Helmut. Über ein metrisches Problem der additiven Zahlentheorie. J. Reine Angew. Math. **197** (1957), 97–103.

Let A be an arbitrary sequence of non-negative integers, α its Schnirelmann density and α^* its asymptotic density. Let B_r be the set of the integers of form rs^2, where r is a positive integer and $s=0, 1, 2, \cdots$. Let γ_r be the Schnirelmann density of the sum $A+B_r$ and γ_r^* its asymptotic density. The author proves the results $\gamma_r\geqq\alpha^4/10r^4$ and $\gamma_r^*\geqq\alpha^{*4}/5r^4$. These bounds are only of interest if they are greater than α and α^*, respectively. For $r=1$, the set B_r is a basic sequence and, unless α is very small, better bounds can be obtained from the theorems on the density of the sum of sequences, one of which is a basic sequence.

A. Brauer (Chapel Hill, N.C.).

Volkmann, Bodo. On uniform distribution and the density of sum sets. Proc. Amer. Math. Soc. **8** (1957), 130–136.

If G is an Abelian semigroup, and d is a non-negative set function defined in G with $d(G)=1$, then two sets A and B contained in G may be said to have the addition property if $d(A+B)\geqq\min\{1, d(A)+d(B)\}$; here, $A+B=\{\text{all } a+b \text{ for } a\in A, b\in B\}$. There are now a number of cases in which a particular semigroup is known to admit a set function d such that the addition property holds for a reasonably inclusive and easily described collection of sets. When $G=I^+$, the non-negative integers, and d is a Schnirelmann-type density, then Mann [Ann. of Math. (2) **43** (1942), 523–527; MR **4**, 35] showed that the addition property holds for all sets A and B which contain 0; this is more often stated in terms of a modified sum $A(+)B=(A+B)\cup A\cup B$. Kneser [Math. Z. **58** (1953), 459–484; MR **15**, 104] also considered I^+, but used the lower asymptotic density, and proved that the addition property holds for all pairs of sufficiently irregular sets A and B. A similar study was made independently by M. Ruchte in his thesis [Univ. of Wisconsin, 1953; see Erdös, Proc. Amer. Math. Soc. **5** (1954), 847–853; MR **16**, 336; Ruchte, Bull. Amer. Math. Soc. **59** (1953), 529–530], with d chosen as the inner arithmetical density. [See Buck, Amer. J. Math. **68** (1946), 568–580; **69** (1947), 413–420; MR **8**, 255, 506.] Macbeath [Proc. Cambridge Philos. Soc. **49** (1953), 40–43; MR **15**, 110] chose G as the k-dimensional torus with d as inner Haar measure, and showed that the addition property held for any pair of measurable sets. The author of the present paper is investigating the addition property with $G=I^k$ and $G=(I^+)^k$, the lattice points and non-negative lattice points in k-space, respectively, and with d as lower asymptotic density. In this context, he is able to verify that the addition property holds for certain highly irregular sets constructed in the following manner. Select irrational numbers $\lambda_1, \lambda_2, \cdots, \lambda_k$ and an open set M, and let $S(\lambda, M)$ be the set of all $n=(n_1, n_2, \cdots, n_k)$ for which $\{\lambda\cdot n\}=(\{\lambda_1 n_1\}, \{\lambda_2 n_2\}, \cdots, \{\lambda_k n_k\})$ lies in M; here, $\{x\}$ is the fractional part of x. Using these sets, he is also able to show that there are no limitations on densities of sum sets, other than those imposed by the addition property: given α_j and γ with $1\geqq\gamma\geqq\sum_1^n\alpha_j$, there exist sets A_j with $d(A_j)=\alpha_j$ and $d(A_1+A_2+\cdots+A_n)=\gamma$.

R. C. Buck.

See also: General Theory of Numbers: Kolberg.

Theory of Algebraic Numbers

★**Nakayama, Tadasi. A conjecture on the cohomology of algebraic number fields and the proof of its special case.** Proceedings of the international symposium on algebraic number theory, Tokyo & Nikko, 1955, pp. 71–76. Science Council of Japan, Tokyo, 1956.

The theorems of the paper reviewed below are conjectured and proved in special cases. Allusion is made to the results which will follow for the cohomology of $K\otimes M$, $J\otimes M$, and $C\otimes M$, where K is an algebraic number field, J its idèle group and C its idèle class group.

J. Tate (Cambridge, Mass.).

Nakayama, Tadasi. Cohomology of class field theory and tensor product modules. I. Ann. of Math. (2) **65** (1957), 255–267.

Let G be a finite group of order n and A a G-module. A is called cohomologically trivial if $H^r(S, A)=0$ for all subgroups S of G and all dimensions r, positive and negative. Theorem 1: If for each prime number p dividing n there exists an integer $r=r_p$ such that $H^r(G_p, A)=H^{r+1}(G_p, A)=0$ for a p-Sylow subgroup G_p of G, then G is cohomologically trivial. Theorem 2: If A is cohomologically trivial and M any G-module, then $A\otimes M$ is cohomologically trivial, provided either A or M is n-torsion free. The proofs are a combination of the following ingenious tricks: (1) A is cohomologically trivial for G if and only if it is so for each Sylow subgroup G_p of G (usual restriction-transfer argument). (2) If $H^r(G, A\otimes M)$ is trivial for one fixed r and all torsion-free finitely generated M, then A is cohomologically trivial (induced representations and dimension shifting). (3) Any M as in (2) is a direct summand of an M' which is a submodule of finite index in an M'' of trivial cohomology (take $M''=M\otimes Z(G)$, $M'=M\otimes Zs+M\otimes I$, where s is the sum of the elements of G and I the augmentation ideal in the group ring $Z(G)$). (4) If G_p is a p-group then any finite G_p-module N has a composition series in which the factors are isomorphic to Z/pZ (because I is nilpotent in $(Z/pZ)(G)$, for example). (5) If the hypothesis of Theorem 1 is true and A is p-torsion free, then $H^r(G_p, A\otimes(Z/pZ))=0$ (consider the exact sequence $0\to A\to A\to A/pA\to 0$. (6) Given any G-module A there exists a torsion-free G-module A' such that $H^r(S, A)\approx H^{r+1}(S, A')$ for all r and S (write A as homomorphic image of a G-free module F and let A' be the kernel).

Theorem 3: If a G-module C and an element $\alpha\in H^2(G,C)$ are such that, for each Sylow subgroup G_p, $H^1(G_p, C)=0$, and $H^2(G_p, C)$ is cyclic of order $(G_p:1)$ generated by the restriction of α to G_p, then G, C is a class formation; when this is so, then the cup product with the restriction of α gives isomorphisms $H^{n-2}(S, M)\approx H^n(S, C\otimes M)$ for all subgroups S and all dimensions n, provided either C or M is torsion free. *J. Tate* (Cambridge, Mass.).

See also: Fields, Rings: Whaples; MacKenzie and Whaples. **General Theory of Numbers:** Chalk. **Analytic Theory of Numbers:** Shimura; van Lint.

Geometry of Numbers

See: Functions of Real Variables: de Rham. **Convex Domains, Integral Geometry:** Blundon; Rogers; Lekkerkerker.

ANALYSIS

Functions of Real Variables

Marcus, Solomon. Sur un théorème de M. A. Marchaud et sur les fonctions dérivables presque partout. C. R. Acad. Sci. Paris **244** (1957), 2345–2347.

A. Marchaud has shown [Fund. Math. **20** (1933), 105–116] that a real function of a real variable that takes on any value at most a finite number of times is derivable almost everywhere and becomes a function of bounded variation if its values are suitably modified on a set of arbitrarily small measure. The proof is indirect. The present note gives a direct proof of two more general theorems.

Let $f(x)$ be a real finite function of the real variable x which has the Darboux property on $[a, b]$. Suppose there exists a set A on $[a, b]$ of zero measure such that each point $\xi\in[a, b]-A$ is isolated in the set $\{x;\ f(x)=f(\xi)\}$. Then $f(x)$ is derivable almost everywhere on $[a, b]$.

If $f(x)$ is derivable almost everywhere on $[a, b]$ then for $\varepsilon>0$, ε arbitrary, there is a set $E\in[a, b]$ with measure less than ε to which corresponds a function $g(x)$ of bounded variation on $[a, b]$, such that $g(x)=f(x)$ on $[a, b]-E$.

R. L. Jeffery (Kingston, Ont.).

Vitushkin, A. G. The relation of variations of a set to the metric properties of its complement. Dokl. Akad. Nauk SSSR (N.S.) **114** (1957), 686–689. (Russian)

The paper deals with further estimates based on the author's definitions of kth variation of functions and sets in Euclidean n-space [On multidimensional variations, Gostehizdat, Moscow, 1955; MR 17, 718]; the value of k most relevant here is $k=0$ and we recall that the kth variation of a set then becomes the number of components. The two main theorems are both concerned with the following situation: we are given a closed subset E of a cubic interval W of side a and we inscribe in $W-E$ a maximal parallel cube whose side we denote by b. We shall quote here only one of the two theorems: it asserts that if p is a number less than n such that the variations of E of dimension greater than p all vanish, and if no translate of a coordinate subspace of dimension greater than or equal to $n-p$ has an intersection with E comprizing more than V_0 components strictly interior to W, then $V_0\geqq(12)^{-n}(a/b)^{n-p}$. *L. C. Young* (Madison, Wis.).

Arnol'd, V. I. On the representability of a function of two variables in the form $\chi[\phi(x)+\psi(y)]$. Uspehi Mat. Nauk (N.S.) **12** (1957), no. 2(74), 119–121. (Russian)

The author considers uniform approximations of the form $\chi[\phi(x)+\psi(y)]$, with χ, ϕ, ψ continuous, to functions $f(x, y)$ that are continuous on a square in the (x, y)-plane. He shows by a simple example that such a function $f(x, y)$ need not admit arbitrarily close approximations in the form considered. He observes also that on the square $[0, 1; 0, 1]$ the function xy is not representable in this form, although it is the uniform limit as $n\to\infty$ of the

function $(x+n^{-1})(y+n^{-1})$, which is so representable: moreover, on this square the latter function is strictly monotonic in each variable, but its limit xy is only broadly monotonic. *H. P. Mulholland* (Exeter).

Lozinski, S. M. Inverse functions, implicit functions and solution of equations. Vestnik Leningrad. Univ. Ser. Mat. Meh. Astr. 12 (1957), 131–142. (Russian. English summary)

The author considers a function f defined and of class C' on an (open) domain G in (real) Euclidean n-dimensional space R^n. Supposing that the Jacobian matrix $J(x)$ of f does not vanish at a point x^0 in G, he establishes (in Theorem 1) the existence, in a certain "spherical" neighbourhood S^* of $\xi^0=f(x^0)$, of a unique inverse function $\phi=f^{-1}$, which is of class C' in S^*. He determines S^* in terms of the function f with the aid of norms, $\|x\|$ for points (vectors) x and $\|A\|$ for matrices A, that satisfy the usual axioms and also the relation $\|Ax\|\leq\|A\|\cdot\|x\|$. He writes r^* for the supremum of numbers r such that $x\in G$ and $J(x)\neq 0$ for $\|x-x^0\|\leq r$, and $D(r)$ for a function, continuous and non-decreasing for $0\leq r<r^*$, such that $\|J^{-1}(x)\|\leq D(r)$ for $\|x-x^0\|\leq r<r^*$. Further, he puts

$$\Delta(r)=\int_0^r \frac{du}{D(u)} \quad (0\leq r\leq r^*),$$

and $\rho^*=\Delta(r^*)$. Then the neighbourhood S^* consists of the points ξ with $\|\xi-\xi^0\|<\rho^*$. The theorem also asserts that $\|\phi(\xi)-x^0\|\leq\Delta^{-1}(\|\xi-\xi^0\|)$ in S^*. A corollary asserts that if $0\leq r<r^*$ and $\|\xi-\xi^0\|\leq r/D(r)$, then there is an x such that $f(x)=\xi$ and $\|x-x^0\|\leq r$. In the case where $\|x\|$ is the ordinary Euclidean norm (or length) of the vector x, this corollary reduces to a theorem proved by A. Ostrowski [Nat. Bur. Standards Appl. Math. Ser. no. 29 (1953), 29–34; MR 15, 190]. The author gives an outline of the proof of his Theorem 1 and also enunciations of theorems on similar lines concerning the solution of the equation $f(x)=\xi$ and the existence of an implicit function.

H. P. Mulholland (Exeter).

Vasilach, Serge. Généralisation d'un théorème de Phragmen. C. R. Acad. Sci. Paris 243 (1956), 1468–1471.

The author proves a theorem which is a generalization to functions of n variables of a result of Phragmén. If $g(x_1, x_2, \cdots, x_n)$ is a continuous function on the set in real n-space defined by $0\leq x_i\leq X_i$, $i=1, 2, \cdots, n$, then

$$\sum_{k_1,\cdots,k_n=1}^{\infty}\prod_{i=1}^{n}\frac{(-1)^{k_i-1}}{k_i!}\int_0^{X_1}\cdots\int_0^{X_n}\exp\Big(\sum_{j=1}^{n}k_jx_j(u_j-\xi_j)\Big)\times g(\xi_1,\cdots,\xi_n)d\xi_1\cdots d\xi_n$$

tends to $\int_0^{u_1}\cdots\int_0^{u_n}g(\xi_1,\cdots,\xi_n)d\xi_1\cdots d\xi_n$ as all the x_i ($i=1, 2, \cdots, n$) tend to infinity. From this he is able to deduce some generalizations to functions of n variables of some results of J. G. Mikusiński and Ryll-Nardzewski [Studia Math. 13 (1953), 51–55; MR 15, 407] concerning functions with bounded moments. *C. E. Langenhop*.

Aruffo, Giulio. Un criterio di compattezza per insiemi di funzioni in più variabili. Ricerche Mat. 4 (1955), 177–190.

The author deals with real-valued functions in Euclidean n-space. Let H be a measurable set in R^n. Denote by $m_r(H)$ the sum of the measures of the projections of H onto the $\binom{n}{r}$ r-dimensional coordinate subspaces of R^n. A sequence $\{u_k\}$ of functions defined in a set I is said to be equi-μ_r-quasi continuous if for any pair ε, γ of positive numbers there exists a decomposition of I into the union of a finite number of sets I_i, $i=1, \cdots, N$, and for each k a set H_k with $m_r(H_k)<\gamma$, such that the oscillation of u_k in I_i-H_k is less than ε. The sequence is equi-μ_r-quasi bounded if for any $\gamma>0$ there is an M_γ such that, for each k, $|u_k(P)|<M_\gamma$ on I except on a set H_k with $m_r(H_k)<\gamma$. Finally, the sequence is μ_r-quasi uniformly convergent in I if for each ε the sequence converges uniformly except on a set H with $m_r(H)<\varepsilon$. If I has finite measure, it is known that a necessary and sufficient condition that $\{u_k\}$ should be compact with respect to μ_r-quasi uniform convergence is that the functions u_k are equi-μ_r-quasi continuous and equi-μ_r-quasi bounded. The author proves, among other things, the following theorem: Let u_k be a sequence of functions of class C^p in a domain D of R^n, satisfying the inequality

$$\int_D \sum_{j_1,\cdots,j_p}^{1,\cdots,n}\left|\frac{\partial^p u_k}{\partial x_{j_1}\cdots\partial x_{j_p}}\right|^m dx_1\cdots dx_n\leq A.$$

Let r be the largest integer, $1\leq r<n$, for which $p-r/m>0$. Suppose that the u_k are equi-μ_r-quasi bounded in D. Then the u_k are equi-μ_r-quasi continuous in D. D is subjected to a smoothness condition; it must be "p-regular in sections". *J. M. Danskin* (Princeton, N.J.).

★ **de Rham, Georges. Sur certaines équations fonctionnelles.** Ecole Polytechnique de l'Université de Lausanne, Centenaire 1853–1953, pp. 95–97. Ecole Polytechnique, Lausanne, 1953.

Certain continuous non-differentiable functions studied by Cesàro [Arch. Math. Phys. (3) 10 (1906), 57–63] and certain singular functions studied by Faber [Math. Ann. 69 (1910), 372–443] and Salem [Trans. Amer. Math. Soc. 53 (1943), 427–439; MR 4, 217] are obtained here elegantly as solutions of the system of functional equations: $f(t/2)=\alpha f(t)$, $f((1+t)/2)=\alpha+(1-\alpha)f(t)$, where α can be either real or complex and such that $|\alpha|<1$, $|1-\alpha|<1$. The author shows: (i) There exists one and only one bounded solution $f(t)$ in $0\leq t\leq 1$; if $t=0.a_1a_2\cdots a_n\cdots$ in the binary scale, $s_0=0$, $s_k=a_1+\cdots+a_k$, then $f(t)=\sum_1^\infty a_k\alpha^k(\alpha^{-1}-1)^{s_{k-1}}$. (ii) $f(t)$ is continuous. (iii) If $|\alpha|>\frac{1}{2}$, $|1-\alpha|>\frac{1}{2}$, then $f(t)$ and its real and imaginary parts nowhere possess a finite derivative. (iv) If α is real, $\alpha\neq\frac{1}{2}$, then $f(t)$ is a real strictly increasing function whose derivative is zero wherever it exists (i.e., almost everywhere). *F. A. Behrend* (Melbourne).

de Rham, Georges. Sur une courbe plane. J. Math. Pures Appl. (9) 35 (1956), 25–42.

The plane curve C_1 is defined as the limit of a sequence of polygons P_n, where P_0 is a polygon of two sides with vertices $A=(-1, 0)$, $B=(1, 0)$, $C=(1, 2)$, and the 2^n+2 vertices of P_n are the points which trisect the $2^{n-1}+1$ sides of P_{n-1}; in particular, P_1 has the vertices $A'=(-\frac{1}{3}, 0)$, $B'=(\frac{1}{3}, 0)$, $C'=(1, \frac{2}{3})$, $D'=(1, \frac{1}{3})$. The curve may be represented in the form $M=M(t)$ $(0\leq t\leq 1)$, by assigning to the midpoints of P_n (which lie on C_1) the values $t=h/2^n$ ($h=0, 1, \cdots, 2^n$) and completing the parametrization by continuity. $M(t)$ satisfies the functional equations

$$(1)\qquad F_0M(t)=M(t/2),\ F_1M(t)=M((1+t)/2),$$

where F_0, F_1 are the affine transformations mapping A, B, C on A', B', C' resp. B', C', D'; and C_1 is the only bounded solution of (1). If $t=0.a_1a_2\cdots a_n\cdots$ is the binary representation of t, then $M(t)=\lim_{n\to\infty}F_{a_1}\cdots F_{a_n}M(0)$. Also, C_1 has a tangent at every t; its gradient is given by

the continued fraction $m(t)=[a_1, 1-a_1, a_2, 1-a_2, \cdots]$. The function $m(t)$, which is essentially the inverse function of Minkowski's function $?m$, nowhere possesses a finite non-zero derivative; since $m(t)$ is monotonic this implies that $m'(t)=0$ (i.e., C_1 has curvature 0) almost everywhere. C_1 has the following arithmetical property: the gradient m at (x, y) is: (i) rational if and only if x, y are rational with common denominator 3^K; (ii) a quadratic irrationality if and only if x, y are rational numbers whose common denominator is not a power of 3.

More generally, the curves $C_\gamma=\lim P_n$ are considered, where the vertices of P_n divide the sides of P_{n-1} in the ratio $1:\gamma:1$. For $\gamma<1$, C has a dense set of angular points; for $\gamma\geqq 1$ it has a tangent at every point, but is non-analytic except for $\gamma=2$ when it is a parabolic arc. It is noted that functional equations similar to (1) characterize curves studied by the author [article reviewed above], von Koch [Ark. Mat. Astr. Fys. **1** (1904), 681–702], Cesàro [Atti Accad. Sci. Fis. Mat. Napoli (2) **12** (1905), no. 15], and Pólya [Bull. internat. Acad. Sci. Cracovie. Cl. Sci. Math. Nat. Sér. A. **1913**, 305–313].

F. A. Behrend (Melbourne).

See also: Measure, Integration: Enomoto; Schaefer. **Functions of Complex Variables:** Potapov.

Measure, Integration

★**Hartman, S.; i Mikusiński, J. Teoria miary i całki Lebesgue'a. [Theory of measure and Lebesgue integral.]** Państwowe Wydawnictwo Naukowe, Warsaw, 1957. 140 pp. zł. 10.

This is a brief introduction to the Lebesgue integral such as might be given in slightly over a dozen lectures. Seven little chapters with suitable diagrams deal with the standard material on the line, using an approach suggested by M. Riesz. This is followed by five further short chapters on the classes L^p, series of orthogonal functions, plane measure, Fubini's theorem and on the definition of the elementary Stieltjes integral. The demand for books of this type appears to the reviewer to be somewhat on a par with that for books entitled 'Calculus made easy'. To the student already weary of mathematical exercises and routines it supplies little of lasting value. To the student ready to enter into a new world of ideas it supplies a meager and commonplace fare. *L. C. Young.*

Dionísio, J. J. Fundamentals of the theory of measure. Rev. Fac. Ci. Univ. Coimbra **25** (1956), 101–173. (Portuguese. English summary)

Expository paper. *K. Krickeberg* (Hamburg).

Fabian, Václav. Measures the values of which are classes of equivalent measurable functions. Czechoslovak Math. J. **7(82)** (1957), 191–234. (Russian summary)

Es sei die σ-Mengenalgebra V der Definitionsbereich eines Wahrscheinlichkeitsmaßes, und n^* das System aller "zufälligen Variablen", d.h. der Klassen fast überall gleicher V-meßbarer reeller Funktionen. Die Arbeit befaßt sich vor allem mit der Integration in bezug auf Maße μ, deren Werte zufällige Variable sind. Zunächst werden Sätze über die Erweiterung eines solchen Maßes μ von einem Ring auf einen σ-Ring bewiesen, wobei das von einem linearen Funktional mit Werten in n^* induzierte äußere Maß eine Rolle spielt. Ist μ definiert auf dem σ-Ring S, so läßt sich das "schwache" Integral $\int f d\mu$ für beliebige nirgends negative S-meßbare Funktionen f so definieren, daß es additiv und unterhalb stetig ist, und es gilt ein Satz über die Darstellung eines linearen Funktionals als schwaches Integral. Bedeutet W eine σ-Unteralgebra von V, so versteht der Verf. unter einem W-Integral ein additives und unterhalb stetiges Funktional, wiederum mit Werten in n^*, das für Funktionen definiert ist, deren Werte W-meßbare Funktionen sind, genauer gesagt, für alle nirgends negativen $(S\times W)$-meßbaren Funktionen, und das folgende Eigenschaften hat: Sind f und g zwei $(S\times W)$-meßbare Funktionen und ist $g(x,\omega)=\bar{g}(\omega)$, so wird $W\int g d\mu=\int \bar{g} d\mu$ und $W\int fg d\mu=\bar{g} W\int f d\mu$, wobei $\bar{g}$ die durch $\bar{g}$ repräsentierte zufällige Variable ist. Bei σ-endlichem μ ist das W-Integral eindeutig; das Problem der Existenz bleibt im allgemeinen offen. μ heißt stark, wenn das V-Integral existiert. Dafür reicht zum Beispiel hin, daß S oder V die σ-Algebra aller Borelschen Mengen eines lokal-kompakten Hausdorffschen Raumes ist, oder daß μ eine bedingte Wahrscheinlichkeit bildet. Beziehungen zum Begriff der bedingten Wahrscheinlichkeit werden eingehend untersucht. Über das W-Integral werden ferner Analoga zum Fubinischen und Radon-Nikodýmschen Satz bewiesen. Existiert das W-Integral, und ist $\nu(A,\omega)$ bei festem ω als Funktion von A ein reelles Maß auf S und bei festem A als Funktion von ω ein Repräsentant der zufälligen Variablen $\mu(A)$, so ist die Funktion $h(\omega)=\int f(\cdot,\omega)d\nu(\cdot,\omega)$ ein Repräsentant der zufälligen Variablen $W\int f d\mu$. {Der Referent weist in diesem Zusammenhang auf die etwas anders gearteten Sätze von D. Maharam [Trans. Amer. Math. Soc. **75** (1953), 154–184; MR **14**, 1071] hin.} *K. Krickeberg* (Hamburg).

Glivenko, E. V. On Hausdorff type measures. Dokl. Akad. Nauk SSSR (N.S.) **112** (1957), 575–578. (Russian)

This paper consists almost entirely of an abridged version of a previous paper by the author [Mat. Sb. N.S. **39(81)** (1956), 423–432; MR **19**, 21]. *H. P. Mulholland.*

Enomoto, Shizu. Dérivation par rapport à un système de voisinages dans l'espace de tore. Proc. Japan Acad. **30** (1954), 721–725.

Let S be the circumference of the perimeter 1 and let $T=S\times S\times S\times\cdots$. Let m be the Lebesgue measure on S and let $\mu=m\times m\times m\times\cdots$ be the product measure on T. It is proved that, under a suitable definition of the derivative $D\psi$ of a set function ψ defined on some subsets of T, the following theorem is true: if f is a function integrable on T, and $\psi(A)=\int_A f d\mu$, then $D\psi=f$ almost everywhere. The definition of the upper derivate $D\psi$ on p. 722 is incorrect since the symbol $\int_I$ is not defined.

R. Sikorski (Zbl **57** (1956), 45).

Cotlar, Mischa. On ergodic theorems. Math. Notae **14** (1955), 85–119 (1956). (Spanish)

Der Verf. beweist die Differentiationssätze von Lebesgue und Hardy-Littlewood und die Konvergenzsätze der Ergodentheorie unter einheitlichen Gesichtspunkten, in ähnlicher Form wie in seiner umfassenderen Arbeit in Rev. Mat. Cuyana **1** (1955), 105–167 [MR **18**, 893]. Betrachtet werden Konvergenz- und Maximalsätze über eine Schar linearer Operatoren T_r, $r=1, 2, \cdots$ oder $0<r<+\infty$, die in jedem L^p mit $1\leqq p<+\infty$ definiert sind und meßbare Funktionen als Werte haben. Im Fall der Differentiationstheorie liegt das Lebesguesche Maß λ in R^n zugrunde, und es ist $T_r f(x)=\lambda(Q(x,r))^{-1}\int_{Q(x,r)} f d\lambda$, wobei $Q(x,r)$ den achsenparallelen Würfel mit dem Mittel-

punkt x und der Kantenlänge r bedeutet. Im Fall der Ergodentheorie hat man ein abstraktes Maß μ nebst einer Gruppe $(\sigma_t)_{-\infty<t<+\infty}$ maßtreuer Transformationen, und es ist $T_r f(x)=r^{-1}\int_0^r f(\sigma_t x)d\mu(t)$. Im Beweis der Theoreme im Fall der Ergodentheorie werden die zuvor für die Differentiationstheorie abgeleiteten benutzt. Der Artikel kann zugleich als bequeme und klare Einführung in die Ergodentheorie dienen; vorausgesetzt werden nur die Grundlagen der Integrationstheorie. *K. Krickeberg.*

Schaefer, Helmut. Einfacher Beweis einer charakteristischen Eigenschaft Riemann-integrierbarer Funktionen. Arch. Math. **8** (1957), 109–111.

Brief elementary exposition of the standard proof that a bounded real-valued function on a finite closed interval I in euclidean n-space is Riemann integrable over I if and only if its discontinuities in I form a set of Lebesgue measure zero. *T. A. Botts* (Charlottesville, Va.).

Cotlar, M.; and Ricabarra, R. On the integral of Caratheodory. Univ. Nac. La Plata. Publ. Fac. Ci. Fisicomat. Serie Segunda. Rev. **5** (1956), 153–161 (1957). (Spanish summary)

A previous paper of the authors under the same title [Mem. Real Acad. Ci. Exact. Fis. Nat. Madrid **4** (1950), 1–47; MR **16**, 228] contains many misprints and some errors. Here the results of this paper are stated in perfected form and some missing proofs are supplied.
H. M. Schaerf (Madison, Wis.).

Deleanu, Aristide. Sur l'intégration des fonctions d'ensemble. Rend. Sem. Mat. Univ. Padova **27** (1957), 27–36.

Let G be an abelian topological group (written additively). Consider a collection $\{x_i : i \in I\}$ of elements of G and, for any finite subset $H \subset I$, write $s_H=\sum x_i$, $i \in H$. This collection of elements is said to be "summable" if the filter generated by the sets $B_H=\{s_K : H \subset K \subset I\}$ converges to an element s in G. The element s is the "sum" of the element $\{x_i\}$ and is written $s=\sum x_i$, $i \in I$. A collection $\{X_i : i \in I\}$ of subsets of G is said to be "summable" if $\{x_i : i \in I\}$ is summable for every choice $x_i \in X_i$. The set of all such sums $\sum x_i$ is denoted by $\sum X_i$, $i \in I$. Next let E be an abstract set and $\mathfrak{J}$ a field of subsets of E. For $A \in \mathfrak{J}$, denote by $\Pi(A)$ the family of all countable partitions π of A into elements of $\mathfrak{J}$. Write $\pi_1 \leqq \pi_2$ if π_2 is finer then π_1. Now consider a function f, defined on $\mathfrak{J}$ and taking as values subsets of G, such that $f(X) \neq \emptyset$ for every $X \in \mathfrak{J}$ and there exists $\pi_f \in \Pi(A)$ so that $\{f(X) : X \in \pi\}$ is summable to a set $s_f(\pi)$ for every $\pi \geqq \pi_f$. Denote by $\mathfrak{F}(A)$ the filter generated by the sets $\{\pi : \pi \geqq \pi_0,\ \pi \in \Pi(A)\}$, for $\pi_0 \in \Pi(A)$, and, for $\Lambda \in \mathfrak{F}(A)$, set $\$_f(\Lambda)=\bigcup_{\pi \in \Lambda} s_f(\pi)$. Then f is said to be integrable on A if the filter of base $\$_f(\mathfrak{F}(A))$ converges to an element of G. This integral contains as special cases integrals considered by the reviewer [Trans. Amer. Math. Soc. **52** (1942), 498–521; MR **4**, 162] and by Ionescu Tulcea [Acad. R. P. Romîne Stud. Cerc. Mat. **5** (1954), 73–142; MR **16**, 805]. *C. E. Rickart.*

Sawashima, Ikuko. Some remarks on the definition of integrals of vector-valued functions. Nat. Sci. Rep. Ochanomizu Univ. **7** (1956), 1–9.

The author adapts a definition of an integral for real-valued functions due to S. Kametani [same Rep. **2** (1951), 6–12; MR **14**, 256] to the case of vector-valued functions. The functions are point functions $x(s)$ defined on a measure space (S, B, m) with values in a convex linear topological space E. The convex hull of a set $A \subset E$ is denoted by $\mathrm{Co}(A)$ and its closure by $\overline{\mathrm{Co}}(A)$. Definition: If there exists a uniquely determined set function $I(x, \sigma)$ with range in E such that (1) $I(x, \sigma) \in m(\sigma)\overline{\mathrm{Co}}(x(\sigma))$ for any $\sigma \in B$, and (2) $I(x, \sigma)$ is a completely additive function of σ, then $x(s)$ is said to be "integrable" over S with respect to m.

Elementary properties of the integral are obtained and it is shown to exist for a certain class of "measurable" functions when E is complete. The relationship of the integral to integrals due to Pettis [Trans. Amer. Math. Soc. **44** (1938), 277–304], R. S. Phillips [ibid. **47** (1940), 114–145; MR **2**, 103] and the reviewer [ibid. **52** (1942), 498–521; MR **4**, 162] is also discussed.
C. E. Rickart (New Haven, Conn.).

Sobolev, S. L. The extensions of abstract function spaces connected with the theory of the integral. Dokl. Akad. Nauk SSSR (N.S.) **114** (1957), 1170–1173. (Russian)

The abstract spaces considered are connected with a Bochner integral modified by the use of particular norms. The author makes a number of extremely pertinent remarks which he illustrates by examples in which the norms are of the usual pth power type. He shows that the completion of the relevant space has to consist in some cases of ideal elements which are not functions of points but set functions and the like. Further comments are promised for the case of norms occurring in the theory of partial differential equations. *L. C. Young.*

Cesari, L.; and Fullerton, R. E. Smoothing methods for contours. Illinois J. Math. **1** (1957), 395–405.

In connection with his work on Cavalieri's inequality and its applications, Cesari devised a first smoothing method for contours [Rend. Mat. e Appl. (5) **15** (1956), 341–365; MR **18**, 882], roughly as follows: Given a map T from the closure of a simply-connected plane domain A, the sum of the continua of constancy of T which meet a set E constitute the corresponding "shaggy set" E'''; further, by the join of two sets in a continuum K, we mean an irreducible subcontinuum which meets both; and the shaggy set which corresponds to this join may be termed shaggy join. This being so, let C be a portion of the boundary of A determined by an interval of prime ends $<2\pi$ between two distinct prime ends ω_1, ω_2, and let E''' be a shaggy join of ω_1, ω_2 in C'''. Then the first smoothing operation consists in replacing C by a corresponding portion of the boundary of the set $A-E'''$. A second and a third smoothing process are now considered, the former of which turns out to be fully equivalent to the first, and the latter partially so.
L. C. Young (Madison, Wis.).

See also: Functions of Real Variables: Marcus. **Harmonic Functions, Convex Functions:** Choquet and Deny. **Topological Vector Spaces:** Kaplan. **Banach Spaces, Banach Algebras, Hilbert Spaces:** Naĭmark; Umegaki. **Numerical Methods:** Hsu.

Functions of Complex Variables

Macintyre, A. J. An overconvergence theorem of G. Bourion and its application to the coefficients of certain power series. Ann. Acad. Sci. Fenn. Ser. A. I. no. 250/23 (1958), 11 pp.

The point z is an e.a.s. (easily approachable singularity)

of the function $f(z)=\sum c_n z^n$ if f can be continued analytically into a domain U: $z=1-re^{i\phi}$ $(0<r<\rho, |\phi|<\alpha$, where $\alpha>\pi/2)$. It is a v.i.s. (virtually isolated singularity) of f if the angle α above can be chosen greater than π (in which case U is no longer schlicht). The author shows that if f is regular in $|z|\leq 1$ except for an e.a.s. at $z=1$, and there exists a positive number γ such that $|c_n|<\varepsilon^{-\gamma n}$ for a sequence of n of positive upper density, then there exists a $\gamma'>0$ such that $\{c_n\}$ has Ostrowski segments in which $|c_n|<\varepsilon^{-\gamma' n}$. He uses this to prove that if f is regular in $|z|\leq 1$, except for a v.i.s. at $z=1$, then for each $\varepsilon>0$ the two inequalities

$$|c_n|>\varepsilon^{-\varepsilon n}, \quad \left|\frac{C_{n+1}}{C_n}-1\right|<\varepsilon$$

hold for all n of a sequence of density 1. *G. Piranian.*

Krasnoščekova, T. I. On zeros of partial sums of a power series. Aviacion. Inst. Sergo Ordžonikidze. Trudy Inst. no. **61** (1956), 37–40. (Russian)

The author applies the theory of normal families to give a proof of Jentzsch's theorem and to derive the following result: Let $\sum a_n z^n$, $\limsup |a_n/a_{n+1}|=1$, denote a power series and $S_n(z)$ the sequence of partial sums. For every number $\delta>0$ the number of zeros of $S_n(z)$ in the domain $|z|\geq 1+\delta$ is a bounded function of n.

H. Tornehave (Copenhagen).

Künzi, Hans; et Wittich, Hans. Sur le module maximal de quelques fonctions transcendantes entières. C. R. Acad. Sci. Paris **245** (1957), 1103–1106.

For certain integral functions which arise from the uniformization of Riemann surfaces whose Speiser graphs have a finite number of periodic ends, it is shown that the maximum modulus is attained either almost on a straight line or on a logarithmic spiral. The authors use a theorem of P. Belinsky, which strengthens the deformation theorem of Teichmüller-Wittich in the theory of quasi conformal mappings. *L. V. Ahlfors.*

Grötzsch, Herbert. Konvergenz und Randkonvergenz bei Iterationsverfahren der konformen Abbildung. Wiss. Z. Martin-Luther-Univ. Halle-Wittenberg. Math.-Nat. Reihe **5** (1955/56), 575–581.

Let B_n be a region of the z-plane which has finite connectivity n and contains $z=\infty$ and $z=z_1$. The boundary components of B_n may be taken to be simple closed analytic curves or analytic arcs. The author gives an iterative procedure for constructing the mapping function f, normalized at $z=\infty$, which takes B_n onto a region bounded by concentric circular slits with center at $f(z_1)$. The process has the advantage that the sequence of approximations converges uniformly to f in all of $\bar{B}_n$; i.e., in B_n and on its boundary. *G. Springer.*

Koritsky, G. V. Curvature of level curves in univalent conformal mappings. Dokl. Akad. Nauk SSSR (N.S.) **115** (1957), 653–654. (Russian)

Let Σ be the class of functions $F(\zeta)=\zeta+\alpha_0+\sum_{n=1}^{\infty}\alpha_n/\zeta^n$ regular and univalent in $1<|\zeta|<\infty$, and let Σ_2 denote the subclass of odd functions. Let K_ρ denote the maximum curvature of the image curve of $|\zeta|=\rho$, $\rho>1$, under $F(\zeta)$. The author proves that for the class Σ and Σ_2

$$K_\rho \leq \frac{\rho(\rho^2+1)}{(\rho^2-1)^2},$$

and the function $\zeta+1/\zeta$ shows that this bound is sharp for both classes. The author also obtains the sharp upper bound for K_ρ in the subclass of functions with p-fold symmetry mapping onto a domain whose complement is starlike with respect to the origin. *A. W. Goodman.*

Komatu, Yûsaku. On conformal mapping of polygonal domains. Proc. Japan Acad. **33** (1957), 279–283.

The author continues earlier studies of the conformal mapping $f(z)$ of a circular ring $q<|z|<1$ onto the exterior of two polygons. He presents a new proof of his generalized Schwarz-Christoffel representation of such mappings in terms of elliptic functions. The basic tools are reflection and analytic continuation, which serve to establish periodicity properties of the single-valued function $F''(w)/F'(w)$, where $F(w)=f(\exp(-iw))$.

P. R. Garabedian (London).

Ahlfors, Lars V. Extremalprobleme in der Funktionentheorie. Ann. Acad. Sci. Fenn. Ser. A. I. no. 249/1 (1958), 9 pp.

The paper, presented at the Helsinki conference on functions of a complex variable, is a summary of postwar developments in the field of extremal problems in conformal mapping. The principal topics reviewed are the method of extremal length, the Hardy class H^p, and the method of interior variation. Emphasis is placed on general principles as the main object of study, rather than special exercises, and it is pointed out that the solutions of most extremal problems in function theory are connected with an appropriate quadratic differential. Finally, the author gives a generalization, based on Riemannian metrics, of the method of interior variation, but he fails to include examples of its application.

P. R. Garabedian, (London).

Jenkins, James A. On the existence of certain general extremal metrics. Ann. of Math. (2) **66** (1957), 440–453.

Let R' be obtained from a finite Riemann surface R by deleting a finite number of points. Let H_i, $i=1, \cdots, k$, be homotopy classes of closed curves on R' or of arcs whose end points are free to move on the contours. A problem closely related to extremal length is the following: Given numbers $a_i\geq 0$, find a metric $\rho|dw|$ which satisfies $\int_{h_i}\rho|dw|\geq a$ for $h_i\in H_i$, and minimizes $\iint_R \rho^2 du dv$.

The existence of such an extremal metric is proved under the condition that there are non-intersecting representations of $h_i\in H_i$. With the extremal mapping there are associated a quadratic differential and a corresponding subdivision of R into annuli and quadrangles, one for each H_i. The result is in accordance with natural conjectures, and the purpose of the paper is to provide a precise proof. Technically, the proof makes strong use of Schiffer's and Spencer's variational method [Functionals of finite Riemann surfaces, Princeton, 1954; MR **16**, 461].

L. V. Ahlfors (Cambridge, Mass.).

★ **Ahlfors, L. V. Remarks on Riemann surfaces.** Lectures on functions of a complex variable, pp. 45–48. The University of Michigan Press, Ann Arbor, 1955. $10.00.

Heuristische Bemerkungen zu einer Verallgemeinerung des klassischen Problems, auf einer geschlossenen Riemannschen Fläche zu vorgegebenen Singularitäten eine harmonische Funktion zu bestimmen. Statt der in den Singularitäten punktierten Fläche wird eine beliebige offene Fläche R genommen, in einer Teilmenge mit kom-

pakten Komplement (einer Umgebung des idealen Randes) ist eine lokal harmonische Funktion s gegeben. Gesucht ist eine auf R überall harmonische Funktion u, so daß $u-s$ eine gewisse Regularitätsbedingung (die verschieden gewählt werden kann) erfüllt.

H. Grunsky (Zbl **66** (1957), 328).

Sario, Leo. Extremal problems and harmonic interpolation on open Riemann surfaces. Trans. Amer. Math. Soc. **79** (1955), 362–377.

Verf. behandelt Interpolationsprobleme für harmonische und analytische Funktionen im Zusammenhang mit bestimmten Extremaleigenschaften auf offenen Riemannschen Flächen. Derartige Überlegungen ergeben sich auf verhältnismäßig einfache Weise für den speziellen Fall einer kompakten Fläche W, die von einer endlichen Anzahl analytischer Jordanbögen begrenzt wird. Hingegen erweist sich die Aufgabe für Flächen mit allgemeinerer Berandung, sowie für solche mit unendlich hohem Geschlecht als bedeutend komplizierter. Verf. ist es gelungen, das folgende Reduktionstheorem aufzustellen, durch welches der allgemeine Fall mit dem zuerst erwähnten spezielleren verbunden wird: Vorausgesetzt, daß das minimalisierende Funktional wachsend ist mit dem Gebiet und die minimalisierenden Funktionen auf kompakten Teilgebieten W_n mit analytischer Berandung eine normale Familie bilden, zieht die Lösbarkeit des Extremalenproblems für W_n diejenige für W nach sich. Im folgenden wird die Existenz gewisser minimalisierender Funktionen mit Hilfe des Dirichletintegrals nachgewiesen.

H. P. Künzi (Zbl **66** (1956), 58).

Kusunoki, Yukio. On Riemann's period relations on open Riemann surfaces. Mem. Coll. Sci. Univ. Kyoto. Ser. A. Math. **30** (1956), 1–22.

Es sei R eine nicht-kompakte Riemannsche Fläche und R_0 ein von endlich vielen Jordankurven berandetes relativ kompaktes Teilgebiet; $\{C\}$ sei ein System von Zyklen C, die homolog ∂R_0 sind und aus endlich vielen analytischen Jordankurven auf $R-R_0$ bestehen. Man weiss, dass die Extremallänge von $\{C\}$ genau dann verschwindet, wenn R nullberandet ist. Das Teilsystem $\{C\}_1 \subset \{C\}$ besteht aus jenen Zyklen C, bei denen jede Komponente R zerlegt. $\{C\}_E$ ist definiert relativ zu einer Ausschöpfung $E=\{R_n\}$ von R: $\{C\}_E=\bigcup_{n=1}^{\infty} L_n$, wo L_n genau aus jenen Zyklen aus $\{C\}_1$ besteht, die in R_n-R_{n-1} verlaufen. Die Flächenklassen O' und O'' sind charakterisiert durch das Verschwinden der Extremallängen $\lambda\{C\}_1$ bzw. $\lambda\{C\}_E$ (für ein E). Eine kanonische Homologiebasis $\{A_n, B_n\}_1^{\infty}$ heisst vom $\mathfrak{A}$-Typus inbezug auf eine Ausschöpfung $\{R_n\}$ [L. Ahlfors, Ann. Acad. Sci. Fenn. Ser. A. I. no. 35 (1947); MR **10**, 28] wenn die A_i und B_i, $i=1, \cdots, k_n$ für die Zyklen auf R_n mod ∂R_n eine Basis sind. Es wird bewiesen:

1. Ist $R \in O'$ und sind φ und ψ zwei abelsche Differentiale erster oder zweiter Gattung (mit nur endlich vielen Polen) und je von endlicher Norm inbezug auf eine Umgebung des idealen Randes, so gibt es eine Homologiebasis vom $\mathfrak{A}$-Typus mit

$$\lim_{n\to\infty} \sum_{j=1}^{k_n} \left(\int_{A_j} \varphi \int_{B_j} \psi - \int_{A_j} \psi \int_{B_j} \varphi \right) = J, \tag{1}$$

wo J nur von den Singularitäten der φ und ψ abhängig ist. Sind dagegen φ und ψ harmonische Formen endlicher Norm, so ist $J=D(\varphi, \psi)$.

2. Ist $R \in O''$, so gibt es eine von den φ und ψ unabhängige Wahl der $\mathfrak{A}$-Basis.

3. Ist $R \in O_{HD}$, sind φ und ψ harmonische Formen und sind „nur endlich viele Perioden von φ oder ψ von null verschieden", so gilt (1) mit $J=D(\varphi, \psi)$ für eine beliebige Homologiebasis.

A. Pfluger (Zürich).

Perry, R. L. Real zeros of integral functions. J. London Math. Soc. **32** (1957), 467–473.

The author investigates the real zeros of

$$f(z, t)=\sum_{n=0}^{\infty} \alpha_n(t) a_n z^n \quad (0 \leq t < 1,\ a_n \geq 0),$$

where $\{\alpha_n(t)\}$ are independent functions whose distribution function $F(u)=P[\alpha_n(t)<u]$ is the same for all n, is symmetric with zero mean, and has a finite non-zero standard deviation. Let $\eta(t, R)$ be the number of zeros of $f(z, t)$ on the real axis between the origin and R. It is shown that $P[\eta(t, R)<A \log\log R] < B(\log R)^{-D}$, and that if $\sum a_n z^n$ is of very regular growth, then $P[\eta(t, R)< A \log R(\log\log R)^{-1}] < R^{-D/\log\log R}$, where A, B, and D are constants with $D>0$. [Regarding the distribution of zeros of certain families of polynomials and integral functions see also Erdös and Offord, Proc. London Math. Soc. (3) **6** (1956), 139–160; MR **17**, 500; and Littlewood and Offord, Proc. Cambridge Philos. Soc. **35** (1939), 133–148 Ann. of Math. (2) **49** (1948), 885–952; **50** (1949), 990–991; MR **10**, 692.]

H. P. Edmundson.

Yurchenko, A. K.; and Dunduchenko, L. E. On the boundary values of functions regular and univalent in the circle $|z|<1$ belonging to certain special classes. Ukrain. Mat. Ž. **9** (1957), 455–460. (Russian)

Let $f(z)=z+\cdots$ be regular and univalent in $|z|<1$ and suppose the image domain is starlike with respect to the origin. Then $f(z)$ can be written in the form

$$f(z)=z \exp\left\{ -\frac{1}{\pi} \int_{-\pi}^{\pi} \ln(1-ze^{-i\theta}) d\mu(\theta) \right\},$$

where $\mu(\theta)$ is a nondecreasing function of bounded variation. The author proves that $f(z)$ will have points of discontinuity on $|z|=1$ if and only if the singular part of $\mu(\theta)$ is not identically zero.

A. W. Goodman.

Umezawa, Toshio. Multivalently close-to-convex functions. Proc. Amer. Math. Soc. **8** (1957), 869–874.

A function $\phi(z)$, analytic for $|z|<1$, is said to be of class $C(p)$ ($p=1, 2, \cdots$) if, for $|z|$ sufficiently close to 1, $G(r, \theta)>0$ and $\int_0^{2\pi} G(r, \theta) d\theta = 2\pi p$, where $G(r, \theta)=1+\Re[z\phi''(z)/\phi'(z)]$. A function $f(z)$, analytic for $|z|<1$, is said to be of class $K(p)$ if there exists a function $\phi(z)$ of class $C(p)$ for which $\mathrm{Re}[f'(z)/\phi'(z)]>0$, $|z|<1$. The author proves that the functions of class $K(p)$ are at most p-valent in $|z|<1$. He obtains a distortion theorem and inequalities for the coefficients a_n, for functions f of class $K(p)$ having the form $z^q+a_{q+1}z^{q+1}+\cdots$, $1 \leq q \leq p$. The proofs are based on earlier papers by the author [Tôhoku Math. J. (2) **7** (1955), 212–228; Proc. Amer. Math. Soc. **3** (1952), 813–820; MR **17**, 1068; **14**, 260] and on a paper by Goodman [Trans. Amer. Math. Soc. **68** (1950), 204–223; MR **11**, 508]. There are several misprints and minor errors.

W. Kaplan (Ann Arbor, Mich.).

Terzioğlu, A. Nazim; und Kahramaner, Suzan. Über das Argument der analytischen Funktionen. Rev. Fac. Sci. Univ. Istanbul. Sér. A. **21** (1956), 145–153 (1957). (Turkish summary)

Let $f(z)=z+\sum_{n=2}^{\infty} a_n z^n$ be regular and univalent in $|z|<1$, and let a_k be the first nonzero coefficient in the sum. Set $F(z)=z/f(z)$. The author studies $|\arg F(z)|$. He

proves the following: (I) if $|\arg F(z)|<\lambda$ for $|z|<1$ then $\lambda \geqq (\pi \log 4)/\log x_0$, where x_0 is the largest root of the equation $|a_k|=(4(x-1)\log 4)/((x+1)\log x)$; and (II) $|\arg F(z)| \leqq |z| \log 16)/(1-|z|^2)$. *A. W. Goodman* (Lexington, Ky.).

Bonami, A. A. On means of moduli of analytic functions. Dokl. Akad. Nauk SSSR (N.S.) **113** (1957), 1195–1198. (Russian)

Let S be a strip $\alpha<x<\beta$ in the $(x+iy)$-plane. Hardy, Ingham and Pólya investigated properties of the means

$$\phi_p(x, y)=(2y)^{-1}\int_{-y}^{y} |f(x+it)|^p dt$$

[Proc. Roy. Soc. London. Ser. A. **113** (1927), 542–569] when f is analytic in the open strip, continuous in the closed strip, and satisfies an order condition. The author considers similar problems when $\log^+|f|$ or $|f|^p$ has a harmonic majorant in S (classes A, H_p), or when $\phi_p(x, y)$ is bounded in S (class $\mathfrak{M}_p$). First he obtains information about the structure of f in terms of Blaschke products, etc. Then he shows, among other things, that if $f \in H_p$, then $\phi_p(x, y) \leqq K$ in S if this is true for $x=\alpha$ and β, and that if $f \in \mathfrak{M}_p$, then $\log \sup_y \phi_p(x, y)$ is convex in x. *R. P. Boas, Jr.* (Evanston, Ill.).

Tumarkin, G. C. On simultaneous approximation in the mean of complex-valued functions given along several closed curves. Dokl. Akad. Nauk SSSR (N.S.) **114** (1957), 710–713. (Russian)

The author has previously considered the space $L^p(d\sigma, \gamma)$ of functions f for which $\int_\gamma |f(\zeta)|^p d\sigma(s)<\infty$ $(p>0)$, where γ is a closed rectifiable Jordan curve [same Dokl. **84** (1952), 21–24; MR **14**, 154]. Now he announces extensions of his investigations to $L^p(d\sigma, \Gamma)$, where Γ is made up of n disjoint curves γ_l bounding a multiply connected region G. If the system $\{\zeta^m\}$, $m=1, 2, \cdots$, is closed in each $L^p(d\sigma_l, \gamma_l)$, then it is closed in $L^p(d\sigma, \Gamma)$. The place of the system of polynomials is taken, in general, by the linear manifold generated by $\{z^n, (z-\alpha_l)^{-m}\}$, where α_l is a point in the component of the complement of $\bar{G}$ bounded by γ_l. For R to be closed in $L^p(d\sigma, \Gamma)$ it is necessary and sufficient that

$$\int_\Gamma \log \sigma'(s)|\Psi'(\zeta)d\zeta|=-\infty,$$

where $w=\Psi(z)$ maps G on a certain canonical region. If this condition does not hold the span of R is characterized. Other results on approximation are given. The author generalizes some results of J. Penez [Proc. Nat. Acad. Sci. U.S.A. **40** (1954), 240–243; MR **15**, 785] and of W. Rudin [Trans. Amer. Math. Soc. **78** (1955), 46–66; MR **16**, 810]. *R. P. Boas, Jr.* (Evanston, Ill.).

Soloviev, A. D. Determination of the class of convergence of interpolation series for certain problems. Dokl. Akad. Nauk SSSR (N.S.) **113** (1957), 991–994. (Russian)

If $\Phi(z)=\sum z^n/m_n$ $(m_n \neq 0, |m_{n+1}/m_n| \to \infty)$, then the integral function $F(z)=\sum c_k z^k$ belongs to class $[\Phi, \sigma]$ if $\limsup_{n\to\infty} |c_n m_n|^{1/n}<\sigma<\infty$. The author extends results of M. A. Evgrafov [Izv. Akad. Nauk SSSR Ser. Mat. **17** (1953), 421–460; MR **15**, 515], which depend on the representation of $F(z)$ in the form

$$F(z)=\frac{1}{2\pi i}\int_C \Phi(z\zeta)f_\Phi(\zeta)d\zeta,$$

and finds the exact $[\Phi, \sigma]$ convergence class for certain interpolation series $\sum A_n(F)p_n(z) \sim F(z)$ with moments

$$A_n(F)=\frac{1}{2\pi i}\int_C \zeta^n \phi_{n,\Phi}(\zeta)f_\Phi(\zeta)d\zeta.$$

His results are then applied to the Abel-Gontcharoff interpolation problem with $A_n(F)=F^{(n)}(\lambda_n)/n!$ in the cases

(i) $|\lambda_n|<|\lambda_{n+1}|$, $\lim_{n\to\infty} \lambda_n/\lambda_{n+1}=q \neq 1$,

(ii) $\lambda_n=\nu_n e^{i\alpha(-1)^n}$, $\nu_n \leqq \nu_{n+1}$, $\nu_n/\nu_{n+1} \to 1$.

(iii) $\lambda_n/\lambda_{n+1} \to 0$.

The exact $[\Phi, \sigma]$ convergence classes for the corresponding interpolation series are found.

A generalisation of the Abel-Gontcharoff problem and two problems in moments, due to A. O. Gelfond [The calculus of finite differences, Gostehizdat, Moscow-Leningrad, 1952; MR **14**, 759], are solved in a similar manner. *S. Macintyre* (Aberdeen).

Evgrafov, M. A.; and Soloviev, A. D. On a class of reversible operators in a ring of analytic functions. Dokl. Akad. Nauk SSSR (N.S.) **114** (1957), 1153–1154. (Russian)

Let K be a ring of analytic functions of m complex variables $Z_1, \cdots, Z_m$, defined and single-valued for $r_k<|Z_k|<R_k$. K is given the topology of uniform convergence on annuli $r_k(1+\varepsilon)<|Z_k|<R_k(1-\varepsilon)$, for any $\varepsilon>0$.

Let λ be any complex number and A_λ a linear operator on K defined by the conditions $A_\lambda Z_1^{n_1}\cdots Z_m^{n_m}=Z_1^{n_1}\cdots Z_m^{n_m}h_{n_1,\ldots,n_m}(Z_1, \cdots, Z_m; \lambda)$, $-\infty<n_1, \cdots, n_m<\infty$. Depending on whether K is considered as a linear space or as a ring, sufficient conditions are given on the function $h_{n_1,\ldots,n_m}$ so that the operator A_λ has a bounded inverse with the exception of a countable number of eigenvalues $\{\lambda_n\}$ which have no finite limit point and each of which has finite multiplicity.

No proofs are given. *A. Devinatz* (St. Louis, Mo.).

Ono, Isao. An analytic kernel in several complex variables. Sci. Rep. Tokyo Kyoiku Daigaku. Sect. A. **5** (1956), 260–266.

For an analytic function $f(z)$ of k complex variables $z=(z_1, \cdots, z_k)$ there exists a familiar Cauchy formula whose kernel is not analytic and depends, indeed, on the distance r from z to the variable point of integration t. The author modifies this formula so that the kernel becomes analytic in z, but the result is a representation which is in general only valid in a subregion of the interior of the surface of integration. The procedure amounts to an analytic continuation of the familiar formula to independent complex values of z and $\bar{z}$, followed by the requirement that $\bar{z}$ should remain fixed while z varies. *P. R. Garabedian* (Stanford, Calif.).

Potapov, M. K. Insertion theorems for analytical functions of many variables. Dokl. Akad. Nauk SSSR (N.S.) **112** (1957), 591–594. (Russian)

The author studies complex-valued L_p-integrable functions $f(x_1, \cdots, x_n)$ of real variables and with period 2π in each variable. A real number $r>0$ is represented as $r'+\alpha$ where r' is an integer and $0<\alpha \leqq 1$. S. M. Nikol'skiĭ [same Dokl. **76** (1951), 785–788; Trudy Mat. Inst. Steklov **38** (1951), 244–278; MR **12**, 603; **14**, 32] introduced the classes $H_p^r(M)$ consisting of L_p-functions $(p>1)$ which possess partial derivatives of order r' with respect to each variable almost everywhere, and such that the r'th

derivatives satisfy a Lipschitz condition of order α and with the constant factor M. The present paper deals with classes $\mathfrak{B}^{\delta_1,\cdots,\delta_n}H_p{}^r(M)$ of functions which are analytic functions of each x_ν when $|\mathrm{Im}\, x_\nu|<\delta_\nu$ while the remaining variables have fixed real values. It is further assumed that the function has limit functions belonging to $H_p{}^r(M)$ when $\mathrm{Im}\, x_\nu \to \pm\delta_\nu$. Let $f \in \mathfrak{B}^{\delta_1,\cdots,\delta_n}H_p{}^r(M)$ and $1\leqq m\leqq n$; $p'>p$. The author proves that the function f for fixed values of $x_{m+1}, \cdots, x_n$ belongs to a class $\mathfrak{B}^{\delta_1,\cdots,\delta_m}H_{p'}{}^{\rho}(M_1)$ where $\rho=r-np^{-1}+mp'^{-1}$. On the other hand, a function of $\mathfrak{B}^{\delta_1,\cdots,\delta_m}H_p{}^{\rho}(M)$ can be extended into a function of $\mathfrak{B}^{\delta_1,\cdots,\delta_n}H_r^p(M_2)$ where $r=\rho+(n-m)p^{-1}$. *H. Tornehave.*

See also: Special Functions: Erwe. **Partial Differential Equations:** Bers. **Complex Manifolds:** Bremermann. **Numerical Methods:** Chao.

Geometric Analysis

See: Manifolds, Connections: Gołab. **Mechanics of Particles and Systems:** Easthope.

Harmonic Functions, Convex Functions

Gabriel, R. M. A result concerning convex level surfaces of 3-dimensional harmonic functions. J. London Math. Soc. **32** (1957), 286–294.

Soit $g(x, y, z)$ harmonique dans un domaine de R^3. On suppose en tout point P: grad $g\neq 0$, la surface $g=g(P)$ d'un coté du plan tangent (du coté du vecteur gradient), ce qui entraîne que la dérivée seconde de g le long de toute tangente en P soit $\leqq 0$ (il semble que cette condition suffise pour la suite). Si en un point P_0, une telle dérivée seconde pour une tangente est nulle, les surfaces de niveau sont des cônes ou cylindres de sommet sur cette tangente. Le cas général où les dérivées secondes considérées sont >0 a lieu en particulier pour la fonction de Green d'un domaine convexe; alors les domaines $G_P>\lambda$ sont convexes comme l'auteur l'avait déjà établi [même J. **30** (1955), 388–401; MR **17**, 358]. On utilise développements et dérivées successives le long de droites.

M. Brelot (Paris).

Gabriel, R. M. Further results concerning the level surfaces of the Green's function for a 3-dimensional convex domain. I. J. London Math. Soc. **32** (1957), 295–302.

On connaît pour les fonctions holomorphes univalentes transformant le cercle-unité en domaine connexe D diverses inégalités que l'auteur transpose en propriétés de la fonction de Green de D. Il les adapte ensuite à la fonction de Green G_P d'un domaine convexe de R^3; par exemple la longueur d'une ligne tangente au gradient à partir du pôle P jusqu'au point M est majorée par $BG_P{}^{-1}(M)$ (B=cte). Puis il étudie spécialement ces lignes, dont la convergence à la frontière établit une correspondance biunivoque entre les points-frontière et ces lignes. C'est un cas particulier d'une étude plus générale sur des familles de domaines convexes et les trajectoires orthogonales des surfaces limitantes. *M. Brelot* (Paris).

Gabriel, R. M. Further results concerning the level surfaces of the Green's function for a 3-dimensional convex domain. II. J. London Math. Soc. **32** (1957), 303–306.

L'auteur a déjà montré que les surfaces de niveau S_k ($G_P=k$) de la fonction de Green G_P d'un domaine convexe de R^3 limitent des domaines convexes et que les courbures des sections normales sont $\neq 0$ [même J. **30** (1955), 388–401; MR **17**, 358; voir aussi l'oeuvre analysé ci-dessus]. Il en tire ici diverses conséquences. Ainsi $D(G_x', G_y', G_z')/D(x, y, z)>0$; la distance de S_k et S_{k+a} est strictement décroissante de k. *M. Brelot.*

Power, G.; and Jackson, H. L. W. A general circle theorem. Appl. Sci. Res. B. **6** (1957), 456–460.

The authors generalize the reviewer's circle theorem [Proc. Cambridge Philos. Soc. **36** (1940), 246–247; MR **1**, 284] as follows.

Let $f_0(z)$ be the complex potential of a distribution lying entirely outside the circle $|z|=a$, and $f_1(z)$ the complex potential of a distribution entirely within the circle. Then $w(z)$ defined by

$$(1)\quad w_e(z)=f_0(z)+A\bar{f}_0(a^2/z)+\int_0^1 F(t)\bar{f}_0(ta^2/z)\,dt + B f_1(z)+\int_1^\infty G(t)f_1(tz)\,dt,\ |z|\geqq a,$$

$$(2)\quad w_i(z)=f_1(z)+C\bar{f}_1(a^2/z)+\int_1^\infty H(t)\bar{f}_1(ta^2/z)\,dt + D f_0(z)+\int_0^1 K(t)f_0(tz)\,dt,\ |z|\leqq a,$$

where $C=-A=1-2\alpha$, $B=2(1-\alpha)$, $D=2\alpha$, $F(t)=K(t)=2\alpha\beta t^{-(1+\beta)}$, $G(t)=H(t)=2(1-\alpha)\beta t^{-(1+\beta)}$, $\alpha=\lambda_1/(\lambda_1+\lambda_2)$, $\beta=(\lambda_3-\lambda_4)/(\lambda_1+\lambda_2)$, has the following properties:

(i) $\nabla^2 \mathrm{R}[w(z)-f_0(z)]=0$, when $|z|\geqq a$,

(ii) $\nabla^2 \mathrm{R}[w(z)-f_1(z)]=0$, when $|z|\leqq a$,

(iii) $\mathrm{R}(w_e)=\mathrm{R}(w_i)$,

$$\mathrm{R}\left(\lambda_1 a\frac{\partial w_e}{\partial z}\right)+\mathrm{R}(\lambda_3 w^e)=\mathrm{R}\left(\lambda_2 a\frac{\partial w_i}{\partial z}\right)+\mathrm{R}(\lambda_4 w_i)$$

when $|z|=a$. *L. M. Milne-Thomson* (Providence, R.I.).

Mařík, Jan. The Dirichlet problem. Časopis Pěst. Mat. **82** (1957), 257–282. (Czech. Russian and English summaries)

A discussion by elementary direct methods of the Dirichlet problem for unbounded regions, the boundary functions being defined only on the set of finite boundary points. The author appears to be unfamiliar with the extensive work of Brelot on the generalized Dirichlet problem, from which the main results of the present paper follow easily. *M. G. Arsove* (Paris).

Choquet, Gustave; et Deny, Jacques. Aspects linéaires de la théorie du potentiel. Théorème de dualité et applications. C. R. Acad. Sci. Paris **243** (1956), 764–767.

Let M and C be the vector spaces of Radon measures and real-valued continuous functions on a locally compact space, and let M_k and C_k be the subspaces consisting of their elements having compact support. The bilinear form $\langle f, \mu\rangle = \int f d\mu$ places C_k and M in duality, as it does C and M_k. A positive linear transformation T of M_k into M which is continuous in the weak topology defined by the duality is called a diffusion mapping, and $T\mu$ is called the potential generated by μ. T^* is then a positive weakly continuous linear transformation of C_k into C. Such a T is said to satisfy "le principe du balayage" if for every open relatively compact set W and every positive measure $\mu \in M_k$ there exists a positive measure $\mu' \in M_k$ supported

by the closure of W such that $T\mu' \leqq T\mu$ with equality holding on W. T^* satisfies "le principe du domination" if $f, g \in C_k$ and $Tf \leqq Tg$ on the support of f imply that $Tf \leqq Tg$. It is shown that these two principles are in duality; that is, T satisfies the first if and only if T^* satisfies the second. Convolution by a measure m on a locally compact group can be regarded as acting either on M_k or on C_k, and it is shown that for such a transformation the two principles are equivalent. Finally, it is shown that in the search for such potential kernels one can assume that m satisfies the cancellation law. The proofs are only sketched. *L. H. Loomis* (Cambridge, Mass.).

See also: Functions of Complex Variables: Ahlfors; Sario; Bonami. **Partial Differential Equations:** Henrici. **Numerical Methods:** Kantorovič, Krylov and Černin.

Special Functions

Koschmieder, Lothar. A generator of orthogonal polynomials in the circle and in the triangle. Rev. Mat. Hisp.-Amer. (4) **17** (1957), 291–298. (Spanish)

The author [Tecnica. Rev. Fac. Ci. Ex. Tec. Univ. Nac. Tucumán **1** (1951), 173–181; Univ. Nac. Tucumán. Rev. Ser. A. **10** (1954), 151–162; MR **16**, 585] presented systems of orthogonal polynomials in simple domains of the plane and of space. Here is calculated a generator function of such polynomials in a circular disk and in a triangular disk. *E. Frank* (Chicago, Ill.).

Macrobert, T. M. Integrals allied to Airy's integrals. Proc. Glasgow Math. Assoc. **3** (1957), 91–93.

Integrals of the types

$$\int_0^\infty \begin{matrix}\cos\\ \sin\end{matrix}(\lambda^n \pm x\lambda^l)\lambda^{k-1}d\lambda$$

are discussed, in which n and l are positive integers with $n>l$ and $\Re k>0$ and which reduce to Airy's integrals for $n=3$, $l=1$, and $k=1$. Their values are sums of generalized hypergeometric functions. *N. D. Kazarinoff.*

Ragab, F. M. Integration of E-functions with regard to their parameters. Proc. Glasgow Math. Assoc. **3** (1957), 94–98.

A short proof of Bateman's formula,

$$\sum_{k=0}^{\infty}[(xy)^{\frac{1}{2}}e^{i\varphi}]^{k-n}\frac{n!}{k!}L_n^{(k-n)}(x)L_n^{(k-n)}(y)=$$

$$\exp[(xy)^{\frac{1}{2}}e^{i\varphi}]L_n[x+y-2(xy)^{\frac{1}{2}}\cos\varphi],$$

is given and the formula is generalized to one involving ultraspherical polynomials in place of $e^{i(k-n)\varphi}$.

N. D. Kazarinoff (Ann Arbor, Mich.).

Carlitz, L. A formula of Bateman. Proc. Glasgow Math. Assoc. **3** (1957), 99–101.

Contour integrals of the Barnes type whose integrands involve products of E-functions and products of Gamma Functions with an E-function are evaluated in terms of E-functions. *N. D. Kazarinoff* (Ann Arbor, Mich.).

Carlitz, L. Some polynomials related to Theta functions. Duke Math. J. **24** (1957), 521–527.

Let $q_0=1$, $q_n=(1-q)(1-q^2)\cdots(1-q^n)$, $n\geqq 1$. Define $\begin{bmatrix}n\\ r\end{bmatrix}=q_n/q_rq_{n-r}$, and put $H_n(x)=\sum_{r=0}^n\begin{bmatrix}n\\ r\end{bmatrix}x^r$. The author proves various identities for these functions, such as

$$(1)\quad H_{m+n}(x)=\sum_{r=0}^{\min(m,n)}(-1)^r q^{\frac{1}{2}r(r-1)}\begin{bmatrix}m\\ r\end{bmatrix}\begin{bmatrix}n\\ r\end{bmatrix}q_r x^r H_{m-r}(x)H_{n-r}(x),$$

$$(2)\quad \prod_{r=0}^{\infty}\frac{1-q^r xyt^2}{(1-q^rt)(1-q^rxt)(1-q^ryt)(1-q^rxyt)}=\sum_{n=0}^{\infty}H_n(x)H_n(y)\frac{t^n}{q_n}.$$

Further identities derivable from these are set down, and, in addition, identities depending on more parameters are proved. The proofs are elementary and depend on application of the operators

$$E^n f(x)=f(q^n x),\quad \Delta^n=(1-E)(q-E)\cdots(q^{n-1}-E).$$

M. Newman (Washington, D.C.).

Al-Salam, W. A. The Bessel polynomials. Duke Math. J. **24** (1957), 529–545.

This is a scholarly account of the formal properties of the Bessel polynomials (B.P.) introduced by Krall and Frink [Trans. Amer. Math. Soc. **65** (1949), 100–115; MR **10**, 453] and defined, in the present author's notation, by

$$Y_n^{(\alpha)}(x)={}_2F_0(-n, n+\alpha+1; -x/2)\quad (n=0, 1, 2, \cdots).$$

After summarizing some known results in this new notation, the author establishes several recurrence relations and states four theorems on the unique characterization of the B.P.'s by recurrence relations. These results parallel some results due to Carlitz [Boll. Un. Mat. Ital. (3) **11** (1956), 371–381; MR **18**, 477] on Jacobi polynomials. The paper then turns to integral representations, mostly in terms of Jacobi polynomials. A large number of generating functions, expansion formulae, addition and multiplication formulae are stated next; many of these are again implied by related results on Laguerre and Jacobi polynomials. Some integral representations for products of B.P.'s and integrals involving B.P.'s conclude the paper.

P. Henrici (Los Angeles, Calif.).

Erwe, Friedhelm. Über gewisse Klassen doppeltperiodischer Funktionen. Acta Math. **97** (1957), 145–188.

The author commences by studying the set $\mathfrak{M}^n$ of functions $f(z)=\sum_{\nu=0}^n f_\nu(z)\bar{z}^\nu$, where the $f_\nu(z)$ are meromorphic in the whole plane; such a study is suggested by the generalised Cauchy-Riemann equation

$$(\partial/\partial x+i\partial/\partial y)^{n+1}f=0.$$

After some results concerning linear dependence and local boundedness, the subject is narrowed to $\mathfrak{E}^n$, the subset of $f\in\mathfrak{M}^n$ which admits a given modulus Ω of periods; in particular to that E^n for which Ω forms a lattice. For this case the author gives representations of $f\in\mathfrak{E}^n$ of the form $f=\sum_{\nu=0}^n u_\nu f_1^\nu$; here f_1 is a particular function in $\mathfrak{E}^1$ but not in $\mathfrak{E}^0$, and the u_ν are meromorphic and unique. One example of a suitable f_1 has the form $\sum_{\omega\in\Omega}(\bar{z}-\bar{\omega})/(z-\omega)^4$; another has the form $b_1\bar{z}+b_2\zeta+\bar{z}$, where ζ is the Weierstrass ζ-function. The set of coefficient-functions f_ν is studied, as are the difference equations which the f_ν must satisfy.

The second half of the paper applies results of the first half to functions $\varrho_n(z)$ defined, for $n\geqq 1$, by $\varrho_n(z)=(n+1)\sum'(\bar{\omega})^n\{(\omega-z)^{-n-2}-\omega^{-n-2}\}$, where the prime (′) serves to exclude $\omega=0$, and where $\varrho_0(z)$ indicates the Weierstrass $\wp$-function. These functions arise in connection with a certain function $\mathfrak{p}_n(z)\in\mathfrak{E}^n$, introduced by C.

Müller. Write also $\wp_0(z)=z^{-2}+\sum_{k=1}^{\infty} c_{0k}z^{2k}$, $\wp_n(z)=\sum_{k=1}^{\infty} c_{nk}z^{2k}$ $(n\geqq 1)$, and $d=27(7c_{02})^2-4(5c_{01})^3$. From the differential equation $\wp_n''=6\sum_{\nu=0}^{n}\wp_\nu\wp_{n-\nu}-10c_{01}$, it is deducible that c_{nk} is expressible as a polynomial in $c_{\nu 1}$, $c_{\nu 2}$, $(\nu=0, \cdots, n)$, with positive rational coefficients. The ultimate aim is to express c_{nk} in terms of c_{01}, c_{02}, c_{11}, c_{12} only. It is shown that $d^n c_{nk}$ is expressible as a polynomial in c_{01}, c_{02}, c_{11}, c_{12} with rational coefficients, which, considered as a polynomial in c_{11} and c_{12} only, is homogeneous and of the nth degree. It is substantially harder to show that this is also true of $d^{n-1}c_{nk}$. This is accomplished for $n=2, 3, 4, 5$; the proofs applies complex-variable methods to the doubly-periodic functions

$$\sum_{\nu=0}^{n}(\nu!)^{-1}\wp_{n-\nu}^{(\nu)}\hat{p}^\nu,$$

where $\hat{p}=-\wp_1/\wp_0'$. There are two applications to special lattices; for example, if $2r\geqq n+2$, then $\sum'_{m_1,m_2}(m_1-im_2)^n(m_1+im_2)^{n-4r}=\alpha_{nr}\omega_1^{4r-2n}\pi^{-n}$, where the α_{nr} are rational, being certainly positive if $n\leqq 3$, and $\omega_1=2\int_0^1(1-\sigma^4)^{-\frac{1}{2}}d\sigma$. Finally it is shown that $\mathfrak{E}^1$, in distinction to $\mathfrak{E}^0$, contains bounded functions.

{Reviewer's remark: While the approach to the functions $\wp_n(z)$ via $\mathfrak{M}^n$ and $\mathfrak{E}^n$ is highly interesting, a more classical approach would give greater generality and simplify some points. Defining ω' as the member corresponding to ω of any second lattice, so that $\omega=m_1\omega_1+m_2\omega_2$, $\omega'=m_1\omega_1'+m_2\omega_2'$, let $\wp_0(z, \lambda)$ be the Weierstrassian function with the periods $\omega+\lambda\omega'$. Then $\wp_n(z)$ is a special case of $\wp_n(z, \lambda)=(-1)^n(n!)^{-1}(\partial/\partial\lambda)^n\wp_0(z, \lambda)$, and the differential equation for $\gamma_n(z)$ can be gotten by applying $(\partial/\partial\lambda)^n$ to that for $\wp_0$. If we put $\wp_n(z, \lambda)=\sum_1^\infty c_{nk}(\lambda)z^{2k}$, $n\geqq 1$, then $(\partial/\partial\lambda)c_{nk}(\lambda)=-(n+1)c_{n+1,k}(\lambda)$. This can be used, for example, to prove inductively the results concerning $d^{n-1}c_{nk}$.} *F. V. Atkinson* (Canberra).

Vinti, John P.; and Leser, Tadeusz. The sums of certain series involving Bessel functions. J. Soc. Indust. Appl. Math. 5 (1957), 15–31.

A number of series of Schlömilch's type, of which

$$\sum_{n=1}^{\infty} n^{-1}J_0(nx)J_0(ny)\sin 2\pi nt$$

may be regarded as typical, are evaluated in terms of elementary functions. The method consists in evaluating an integral involving a saw-tooth function in two different ways. *P. Henrici* (Los Angeles, Calif.).

Shapiro, Victor L. Localization on spheres. Trans. Amer. Math. Soc. 86 (1957), 212–219.

Let Ω denote the unit $(k-1)$-dimensional sphere in Euclidean k-space, $k\geqq 3$. Let $S=\sum_{n=1}^{\infty} y_n(x)$ be a series of surface spherical harmonics defined on Ω with $y_n(x)=O(n^\beta)$, $\beta\geqq 0$, uniformly for x on Ω. Let

$$F(x)=\sum_{n=1}^{\infty}(-1)^w y_n(x)[n(n+p)]^{-w},$$

where w is the smallest integer greater than $\frac{1}{2}(\beta+1)$, and $p=k-2$. The author proves that if the function F is of class C^{2w} on a domain D of Ω, then the series S is uniformly summable by Cesàro means of order α, $\alpha>p+2w$, to the sum $\Delta^w F(x)$, in any domain whose closure is contained in the interior of D. Here Δ^w denotes the Laplacian of order w on Ω. The result is an extension of Riemann's classical principle of localization to higher dimensions [cf. A. Zygmund, Trigonometrical series, Warsaw-Lvov, 1935, p. 286]. *K. Chandrasekharan.*

Chatterjee, S. K. On certain definite integrals involving Legendre's polynomials. Rend. Sem. Mat. Univ. Padova 27 (1957), 144–148.

The integral

$$\int_{-1}^{1}\frac{d^r P_m(x)}{dx^r}\frac{d^s P_n(x)}{dx^s}\,dx$$

is evaluated in terms of binomial coefficients by use of Rodriguez' formula. *P. Henrici* (Los Angeles, Calif.).

Erdélyi, A.; and Swanson, C. A. Asymptotic forms of Whittaker's confluent hypergeometric functions. Mem. Amer. Math. Soc. no. 25 (1957), 49 pp. $1.40.

This paper gives a very clear and detailed investigation of the asymptotic expansions of the Whittaker functions $M_{k,m}(z)$ and $W_{k,m}(z)$, when k is large, real or complex, m is fixed and z/k is real. Two pairs of asymptotic forms, uniform in $x=z/4k$ as $k\to\infty$, are obtained which together cover the whole real x axis. These results are deduced from a study of the differential equation

$$\frac{d^2y}{dx^2}+[4k^2(x^{-1}-1)-(m^2-\tfrac{1}{4})x^{-2}]y=0,$$

without using integral representations or other elaborate properties of the functions.

The first pair, involving Bessel functions, are

$$M_{k,m}(4kx)=2^{\frac{1}{2}}\Gamma(2m+1)k^{\frac{1}{2}-m}\{(\psi(x)/\psi'(x))^{\frac{1}{2}}J_{2m}(4k\psi(x))+O[k^{-1}x^{\frac{1}{2}}(1+|x|)^{-\frac{1}{4}}\tilde{U}_0(x)]\},$$

for $-\infty<x<1$, $|k|$ large, $-(1+\frac{1}{4}j)\pi\leqq\arg k\leqq(1-\frac{1}{4}j)\pi$ when $\arg k=j\pi$, $(j=0, \pm 1)$; and

$$W_{k,m}(4kx)=2\pi^{\frac{1}{2}}k^{\frac{1}{2}+k}e^{-k}\{(\psi(x)/\psi'(x))^{\frac{1}{2}}[\sin\pi(k-m)J_{2m}(4k\psi(x))-\cos\pi(k-m)Y_{2m}(4k\psi(x))]+O(k^{-1}\tilde{U}_0(x))+O(k^{-1}U_1(x))\},$$

for $-\infty<x<1$, $|k|$ large, $|\arg k|\leqq\frac{1}{2}\pi$; where $\psi(x)=\frac{1}{2}(x-x^2)^{\frac{1}{2}}+\frac{1}{2}\sin^{-1}(x^{\frac{1}{2}})$;

$$\tilde{U}_0(x)=[\psi(x)/\psi'(x)]^{\frac{1}{2}}J_{2m}(4k\psi(x)),$$

if $|4k\psi(x)|\geqq\delta$, $|\arg(4k\psi(x))|\leqq\varepsilon$ and $|\arg(-4k\psi(x))|\leqq\varepsilon$;

$$\tilde{U}_0(x)=[\psi(x)\{|J_{2m}(4k\psi(x)^2|+|Y_{2m}(\pm 4k\psi(x))|^2\}/\psi'(x)]^{\frac{1}{2}},$$

if $|4k\psi(x)|\geqq\delta$ and $|\arg(\pm 4k\psi(x))|\leqq\varepsilon$; and

$$U_1(x)=\{\psi(x)\}^{\frac{1}{2}}H_{2m}^{(1)}(4k\psi(x))/\{\psi'(x)\}^{\frac{1}{2}}.$$

The second pair, involving Airy functions, are

$$M_{k,m}(4kx)=2^{\frac{1}{2}}\Gamma(2m+1)k^{\frac{1}{2}-m}e^{\pm(\frac{1}{2}+m-k)\pi i}\{-(\phi'(x))^{-\frac{1}{2}}\times Ai[-(4k)^{\frac{2}{3}}\omega^{\pm 1}\phi(x)]+O[k^{-1}V_{\pm 1}(x)]\},$$

for $0<x<\infty$, $|k|$ large, $|\arg k|<\pi$, $\omega=e^{2\pi i/3}$ (upper signs for $0<\arg k<\pi$, lower signs for $-\pi<\arg k<0$, and the sum of both expressions when $k>0$); and

$$W_{k,m}(4kx)=2^{\frac{1}{2}}\pi^{\frac{1}{2}}k^{\frac{1}{2}+k}e^{-k}\{(-\phi'(x))^{-\frac{1}{2}}Ai[-(4k)^{\frac{2}{3}}\phi(x)]+O[k^{-1}x^{-1}\tilde{V}_0(x)]\}$$

for $0<x<\infty$, $|k|$ large and $|\arg k|\leqq\frac{1}{2}\pi$; where

$$\phi(x)=\begin{cases}\left\{\frac{3}{2}\int_x^1\left(\frac{1}{t}-1\right)^{\frac{1}{2}}dt\right\}^{\frac{2}{3}}, & \text{for } 0\leqq x\leqq 1,\\ -\left\{\frac{3}{2}\int_1^x\left(1-\frac{1}{t}\right)^{\frac{1}{2}}dt\right\}^{\frac{2}{3}} & \text{for } 1\leqq x<\infty,\end{cases}$$

$$\tilde{V}_r(x)=\begin{cases}V_r(x), \text{ if } |\arg(-(4k)^{\frac{2}{3}}\omega^r\phi(x))|<\pi,\\ (-\phi'(x))^{-\frac{1}{2}}[|Ai(-(4k)^{\frac{2}{3}}\omega^r\phi(x))|^2\\ \quad+|Bi(-4k)^{\frac{2}{3}}\omega^r\phi(x))|^2]^{\frac{1}{2}}, \text{ if } |\arg((4k)^{\frac{2}{3}}\omega^r\phi(x))|<0,\end{cases}$$

$V_r(x)=[-\phi'(x)]^{-\frac{1}{2}}Ai[-(4k)^{\frac{2}{3}}\omega^r\phi(x)]$, for $r=0, \pm 1$.

{The reviewer has recently completed an independent investigation of these asymptotic forms (Chap. IV, Confluent hypergeometric functions, Cambridge University Press; in course of publication), and obtained similar results.} *L. J. Slater* (Cambridge, England).

Kreyszig, Erwin. On the zeros of the Fresnel integrals. Canad. J. Math. 9 (1957), 118–131.

The paper is concerned with the Fresnel integrals $C(z)=\int_0^z t^{-\frac{1}{2}}\cos tdt$, $S(z)=\int_0^z t^{-\frac{1}{2}}\sin tdt$, in the complex z-plane ($z=x+iy$). After stating some classical relations to other functions, the asymptotic expansions of $C(z)$ and $S(z)$ for $z\to\infty$, $|\arg z|\leqq\frac{1}{2}\pi-\varepsilon$ and $|\arg z|\geqq\frac{1}{2}\pi+\varepsilon$ are developed. Turning to the investigation of the zeros, the author proves the theorem: $C(z)$ cannot possess zeros in any of the strips parallel to the y-axis corresponding to the values $0<x\leqq\pi$, $(4n-1)\pi/2\leqq x\leqq(2n+1)\pi$ $(n=1, 2, \cdots)$. The same is true for $S(z)$ with respect to the strips $0<x\leqq 3\pi/2$, $2n\pi\leqq x\leqq(4n+3)\pi/2$ $(n=1, 2, \cdots)$. Using the first terms of the asymptotic expansion, simple approximate expressions are derived for the real and imaginary part of the zeros of $C(z)$ and $S(z)$ in the remaining strips parallel to the y-axis. These formulas lead to the result: The zeros of $C(z)$ and $S(z)$ are asymptotically located on the curve $y=\pm\frac{1}{2}\log 2\pi x$ in alternating order. For the real and imaginary part of the smallest zero of $C(z)$ the degree of accuracy of these first approximative formulae amounts to about 2 per cent and 1 per cent, respectively. The accuracy increases with increasing values of $|z|$. Some remarks on a more exact determination of the zeros are made and a table of the first 50 zeros of both functions in the first quadrant is given to two decimal places.

W. C. Rheinboldt (Washington, D.C.).

See also: Analytic Theory of Numbers: Shimuro; van Lint. **Approximations, Orthogonal Functions:** Suetin; Chester, Friedman and Ursell. **Integral Transforms:** Rooney. **Ordinary Differential Equations:** Belkina. **Difference Equations, Functional Equations:** Ghermănescu.

Sequences, Series, Summability

Hewitt, Edwin; and Williamson, J. H. Note on absolutely convergent Dirichlet series. Proc. Amer. Math. Soc. 8 (1957), 863–868.

Applying the theory of l_1-algebras of commutative semigroups [E. Hewitt and H. S. Zuckerman, Trans. Amer. Math. Soc. 83 (1956), 70–97; MR 18, 465], the authors obtain the following result: Let $s=\sigma+it$ and $f(s)$ be represented for $\sigma\geqq 0$ by (1) $\sum_{n=1}^{\infty} a_n n^{-s}$, where $\sum_{n=1}^{\infty}|a_n|<\infty$. Then $|f(s)|\geqq k>0$ for $\sigma\geqq 0$ implies that $(f(s))^{-1}$ is also of the form (1), i.e., $(f(s))^{-1}=\sum_{n=1}^{\infty} b_n n^{-s}$ for $\sigma\geqq 0$, where $\sum_{n=1}^{\infty}|b_n|<\infty$. A corresponding result is given for series of the form $\sum_{r\in Q} a_r r^{-it}$, where Q denotes the multiplicative group of positive rational numbers.

H.-E. Richert (Göttingen).

Melentsov, A. A. A contribution to the theory of Hausdorff transformations. Dokl. Akad. Nauk SSSR (N.S.) 113 (1957), 501–502. (Russian)

The author calls the triangular matrices $\|C_{mn}\|$ and $\|C_{mn}{}^*\|$ $(m,n=0, 1, 2, \cdots)$ allied if $C_{mn}{}^*=C_{m,m-n}$ for $n=0, 1, \cdots, m$ and $m=0, 1, 2, \cdots$. He then states several propositions concerning the matrices associated with the Hausdorff, the Nörlund, and the "Summierungsfunktion" [O. Perron, Math. Z. 18 (1923), 157–172] methods of summability: The matrix allied with a Hausdorff matrix is a Hausdorff matrix (in fact, if $C_{mn}=\binom{m}{n}\Delta^{m-n}\mu_n$, then $C_{mn}{}^*=\binom{m}{n}\Delta^{m-n}\mu_n{}^*$ with $\mu_n{}^*=\Delta^n\mu_0$); the necessary and sufficient condition that the Nörlund matrix $\|p_n/P_m\|$ be a Cesàro (Hausdorff) matrix is that $p_n=p_1P_{n-1}/(np_0)$; the necessary and sufficient condition that the matrix allied with a Summierungsfunktion matrix be a Summierungsfunktion matrix is that the latter be an Euler matrix. All of his results can be painlessly deduced by easy calculations and/or from definitions and theorems in Hardy, "Divergent series" [Oxford, 1949; MR 11, 25].

A. E. Livingston (Seattle, Wash.).

Koçak, Cevdet. Die Summierung divergenter Reihen durch analytische Fortsetzung mittels der Theorie der Differenzengleichungen. Bull. Tech. Univ. Istanbul 9 (1956), 43–57. (Turkish summary)

Zur Summierung von Reihen $\sum a(n)$ wird vorgeschlagen, den $\lim \sum_n a(z+nh)$ $(h\to 1, z\to 0; h$ geeignet komplex) oder allgemeiner, falls diese Reihe nicht konvergiert, den $\lim D^{-1}\sum_n Da(z+nh)$ für einen passenden Differentialoperator zu betrachten. Verf. bemerkt, S. 47: "Dass die Methode mehr leisten muss als alle bisherigen, geht schon daraus hervor, dass sich jede beliebige andere Summationsmethode durch Überlagerung der neuen Methode erweitern und verbessern lässt."

A. Peyerimhoff (Giessen).

Hirokawa, Hiroshi. Riemann-Cesàro methods of summability. II. Tôhoku Math. J. (2) 9 (1957), 13–26.

Pour les notations et la définition du procédé de Riemann-Cesàro (R, p, α), voir la première partie de cette note [même J. 7 (1955), 279–295; MR 17, 1076]. Ici, l'auteur considère quelques théorèmes de nature tauberiennes rélatifs à ce procédé de sommation. Soit p un entier positif; de $s_n{}^p=o(n^\gamma)$, $0<\gamma<\beta$, et $\sum_n^\infty \nu^{-1}|a_\nu|=O(n^{-1+\delta})$, $0<\delta<1$, $\delta=p(\beta-\gamma)/(\beta+1-p)$, résulte que $\sum_1^\infty a_n$ est (R, p, α)-sommable vers zéro pour $-1\leqq\alpha\leqq 0$. Dans le cas $\beta=p-1$, les conditions $\sum_1^n |s_\nu{}^{p-1}|=o(n^p/\lg n)$, $\sum_n^{2n}(|a_\nu|-a_\nu)=O(n^{p-\delta})$, $0<\delta<1$, assurent la sommabilité (R, p, α) de $\sum a_\nu$ $(-1\leqq\alpha\leqq 0)$.

M. Tomić.

See also: Analytic Theory of Numbers: Maass. **Functions of Complex Variables:** Perry; Soloviev.

Approximations, Orthogonal Functions

Tandori, Károly. Über die orthogonalen Funktionen. I. Acta Sci. Math. Szeged 18 (1957), 57–130.

D. Menchoff [Fund .Math. 4 (1923), 82–105] and H. Rademacher [Math. Ann. 87 (1922), 112–138] have shown that if $\{a_n\}_0^\infty$ is a real sequence for which $\{a_n \ln n\}_2^\infty\in l^2$ and if $\{\phi_n\}_0^\infty$ is an ONS for a finite interval I, then $\sum a_n\phi_n(x)$ converges a.e. on I. Menchoff showed further that this result is best possible in the sense that if $0<W(n)=o(\ln n)$ and if I is a finite interval, then there exists a uniformly bounded ONS $\{\Phi_n\}_0^\infty$ for I and a real sequence $\{a_n\}_0^\infty$ such that $\{a_nW(n)\}_0^\infty\in l^2$ and $\sum a_n\Phi_n(x)$ diverges on I. Using an argument similar to that of Menchoff, the author of the paper here being reviewed proves that if $\{a_n\}_0^\infty$ is a positive, nonincreasing sequence of real numbers for which $\{a_n \ln n\}_2^\infty\in l^2$, and if I is a finite interval, then there exists a uniformly bounded ONS $\{\Phi_n\}_0^\infty$ for

I such that $\sum a_n\Phi_n(x)$ is everywhere divergent on I; he shows also that this result includes Menchoff's theorem above.

If $\{\phi_n\}_0^\infty$ is an ONS for a finite interval I, and if $\{\mu_n\}_0^\infty$ is a positive, non-decreasing sequence of real numbers, then the Menchoff-Rademacher theorem above, in conjunction with a lemma of Kronecker, allows the author to deduce (1) if $\{1/\mu_n\}_0^\infty \in l^2$, then $\sum_0^n \phi_k^2(x)=o(\mu_n^2)$ for a.a. x in I and (2) if $\{\mu_n^{-1}\ln n\}_2^\infty \in l^2$, then $\sum_0^n \phi_k(x)=o(\mu_n)$ a.e. on I. These results improve known estimates of S. Kaczmarz [Studia Math. **1** (1929), 87–121] and Rademacher, respectively, and the author shows that they are best possible over the class of all ONS on I. He proceeds in the same vein to improve on some estimates of Kaczmarz for the Lebesgue functions for ordinary convergence and of I. S. Gál [Ann. Inst. Fourier, Grenoble **1** (1949), 53–59; MR **12**, 405] and G. Alexits [Ann. Soc. Polon. Math. **25** (1952), 183–187; MR **14**, 1081] for the Lebesgue functions for (C, α), $\alpha>0$, summability, showing in each case that his estimates are best possible in the class of all ONS on I.

A. E. Livingston (Seattle, Wash.).

Suetin, P. K. On polynomials orthogonal along a smooth boundary with differentiable weight. Dokl. Akad. Nauk SSSR (N.S.) **114** (1957), 498–501. (Russian)

The author characterizes the positive weight function $n(z)$ defined on the boundary Γ of a bounded domain G by $n^{(p)}(z) \in \mathrm{Lip}\,\alpha$, $\alpha<1$, and the smoothness of Γ by $\Phi^{(p+1)}(w) \in H_1$, where $z=\Psi(w)$ is the analytic function inverse to $w=\Phi(z)$ mapping G_∞, the complement of $\bar{G}$, onto $|w|>1$ with the normalization $\Phi(\infty)=\infty$, $\Phi'(\infty)>0$. The function $D(w)$, analytic in $|w|>1$ with boundary values representing the weight function $n[\Psi(w)]=|D(w)|^2$ [cf. G. Szegö, Orthogonal polynomials, Amer. Math. Soc. Colloq. Publ., v. 23, New York, 1939, Ch. 10, 16; MR **1**, 14], has a pth derivative in Lip α for $|w|\geqq 1$, as does $g[\Psi(w)]=[\sqrt{(\Psi'(w))}D(w)]^{-1}$. The boundary condition intervenes via the lemma: With $|w_0|=1$,

$$F(w, w_0)=\frac{\Psi'(w)}{\Psi(w)-\Psi(w_0)}-\frac{1}{w-w_0}$$

is $p+1$ times continuously differentiable in $|w|\geqq 1$. The author extends his earlier results [same Dokl. (N.S.) **106** (1956), 788–791; MR **18**, 33] on the asymptotic behavior of the polynomials $\{P_n(z)\}$ orthogonal with respect to $n(z)$ on Γ: With F closed and C depending upon F alone, (i) $P_n(z)\leqq C(F)n^{-p-\alpha}$, $z\in F\subset G$; (ii) $P_n(z)=g(z)[\Phi(z)]^n[1+o(n^{-p-\alpha+1})]$, $z\in\Gamma$; and (iii) $P_n(z)=g(z)[\Phi(z)]^n[1+o(n^{-p-\alpha})]$, $z\in F\subset G$. An expansion theorem for functions with pth primitive in H_1 follows from earlier arguments [same Dokl. (N.S.) **109** (1956), 36–39; MR **18**, 387]. The proofs are based upon the connexion between $P_n(z)$ and the generalized Faber polynomial, $B_n(z)$,

$$\sum_{n=0}^{\infty}\frac{B_n(z)}{w^{n+1}}=\frac{g[\Psi(w)]\Psi'(w)}{\Psi(w)-z},$$

$z\in G$, $|w|>1$, and the relations arising in establishing analogous results for the $\{B_n(z)\}$. [For the argument as to the expansion in the $\{B_n(z)\}$ cf. Suetin, ibid. **88** (1953), 25–28; MR **14**, 740.]

G. Crane (Pittsburgh, Pa.).

Kazmin, Yu. A. On complete systems and bases in L_2. Dokl. Akad. Nauk SSSR (N.S.) **113** (1957), 1199–1202. (Russian)

Let L_2 be the space of square integrable functions on the interval $[a, b]$. A system $\{g_n\}\subset L_2$ is said to be minimal if no one element of $\{g_n\}$ is contained in the closure of the linear manifold generated by the others. It is strongly minimal if there exists a fixed $\delta>0$ such that the distance of any g_n to the closure of the linear manifold generated by the other elements of $\{g_n\}$ is greater than δ. Two systems $\{g_n\}$ and $\{f_n\}$ are called "quadratically-close" if $\sum_1^\infty \|R_n\|^2<\infty$, $R_n=f_n-g_n$, "B-close" if $\sum_{j,k=1}^\infty |(R_j, R_k)|<\infty$.

A representative theorem is the following: If $\{g_n\}$ is a complete, bounded, strongly minimal system, and $\{f_n\}$ is a minimal system, B-close to $\{g_n\}$, then $\{f_n\}$ is complete in L_2.

A. Devinatz (St. Louis, Mo.).

Walsh, J. L.; and Motzkin, T. S. Polynomials of best approximation on a real finite point set. Proc. Nat. Acad. Sci. U.S.A. **43** (1957), 845–846.

For a finite real point set E: $(x_1, \cdots, x_m)$ the authors introduce the concept of a juxtapolynomial $p_n(x)$, of degree n, to a function $f(x)$ on E. This means that there is no polynomial $q_n(x)$ of degree n such that $|f(x)-q_n(x)|\leqq |f(x)-p_n(x)|$ on E, with equality only when $f(x)=p_n(x)$. Taking $m\geqq n+2$, the first result identifies the class of juxtapolynomials with two other classes, that for which $f(x)-p_n(x)$ has at least $n+1$ "weak sign changes", and the $p_n(x)$ of best approximation to $f(x)$ in the sense of least first powers with suitable positive weights. Two similar results involve juxtapolynomials on subsets of E, and approximation in terms of least pth powers with $p\neq 1$. The nature of the set of juxtapolynomials, for given $f(x)$ and $m\geqq n+2$, is also discussed. Proofs are to appear elsewhere.

F. V. Atkinson (Canberra City).

Morozov, M. I. On the question of approximation of periodic quasi-smooth functions, and functions satisfying a Lipschitz condition. Aviacion. Inst. Sergo Ordžonikidze. Trudy Inst. no. 61 (1956), 41–57. (Russian)

The class in question is $\tilde{H}^{(\alpha)}$, consisting of continuous functions of period 2π for which

$$|f(x+h)+f(x-h)-2f(x)|\leqq K|h|^\alpha$$

for some K; here $0<\alpha\leqq 1$, and the class is the same as Lip α for $\alpha<1$. Let f have Fourier coefficients a_n, b_n; the author associates with f sequences of trigonometric polynomials U_n of the form $\frac{1}{2}a_0+\sum_{k=1}^n \mu_k^{(n)}(a_k\cos kx+b_k\sin kx)$ and investigates the maximum $\rho_n^{(\alpha)}$ over $\tilde{H}^{(\alpha)}$ of $\max_x |f(x)-U_n(x)|$. First he proves a general theorem: If $\frac{1}{2}+\sum_{k=1}^n \mu_k^{(n)}\cos ku\geqq 0$ and $J(x;\alpha)=\int_0^x t^{\alpha-1}\sin t dt$, then

$$\rho_n^{(\alpha)}=\tfrac{1}{2}\pi^\alpha/(1+\alpha)-\alpha\pi^{-1}\sum_{k=1}^n \mu_k^{(n)}J(k\pi;\alpha)k^{-\alpha-1}.$$

He applies this when $(\mu_k^{(n)})$ is the matrix of $(C, 1)$ summability, Jackson summability [Safronova, Dokl. Akad. Nauk SSSR (N.S.) **73** (1950), 277–278; MR **12**, 94], and de la Vallée-Poussin summability, in each case obtaining more explicit inequalities for $\rho_n^{(\alpha)}$. Extensions to functions of two variables are outlined.

R. P. Boas, Jr.

Brudny, Yu. A.; Hopenhaus, I. E. On a problem raised by N. N. Lusin. Dokl. Akad. Nauk SSSR (N.S.) **113** (1957), 12–15. (Russian)

Let f be a continuous function on the closed unit interval and let f_n be the polynomial of degree at most n that is the best uniform approximation to f. Let M_n be the subset on which $f-f_n$ attains its maximum modulus. The authors exhibit an f for which M_n is of positive measure for infinitely many n, thereby answering Question 28 from the list in Lusin's book "Integral and trigonometric series" [Gostehizdat, Moscow-Leningrad, 1951; MR **14**, 2].

An obvious modification of the construction yields a parallel example for an f on the unit circle uniformly approximated by trigonometric polynomials f_n. On the other hand, their Th. 3 implies that if the various differences f_n-f_k can be constant only on sets of measure zero (and this will be so whenever the approximating functions f_n are analytic) then the measure of M_n at least approaches zero. The proof is valid only in the real case, however, since Th. 3 is false for complex f_n. Counterexample: $f_n(x)=ne^{ix}$. *H. Mirkil* (Hanover, N.H.).

Chester, C.; Friedman, B.; and Ursell, F. An extension of the method of steepest descents. Proc. Cambridge Philos. Soc. **53** (1957), 599–611.

Let $g(z)$, $f(z, \alpha)$ be analytic functions of their arguments, and let N be a large positive parameter. Asymptotic expansions for integrals of the form

$$\int g(z)\exp[Nf(z, \alpha)]dz$$

can frequently be found by the method of steepest descents, which takes into account the fact that the principal contribution to the integral arises from the points where $\partial f/\partial z=0$, the so-called "saddle-points". The position of the saddle point varies with α, and if for some α (say $\alpha=0$) two saddle points coincide (say at $z=0$), the ordinary method of steepest descents gives expansions which are not uniformly valid near $\alpha=0$. In the paper under review uniform asymptotic expansions are established. The method consists in introducing a new variable $u=u(z, \alpha)$ by the relation

$$f(z, \alpha)=\tfrac{1}{3}u^3-Z(\alpha)u+A(\alpha),$$

where $Z(\alpha)$, $A(\alpha)$, and the branch of u are determined by the condition that the transformation $z\to u$ is (1, 1) near $\alpha=0$. The resulting expansions are ordinary asymptotic expansions (with coefficients depending on α), multiplied by Airy functions of argument $N^{\frac{1}{3}}Z$. The application of the method to Bessel functions of large order is outlined; the results in this case are less general than those obtained by Olver [Philos. Trans. Roy. Soc. London. Ser. A. **247** (1954), 328–368; MR **16**, 696]. *P. Henrici.*

Franklin, Joel; and Friedman, Bernard. A convergent asymptotic representation for integrals. Proc. Cambridge Philos. Soc. **53** (1957), 612–619.

Let $f(x)$ be of class C^{2N} for $x>0$, and let $|f^{(h)}(x)|\leq Me^{\mu x}$, $x>0$ ($h=0, 1, \cdots, 2N$) for suitable M and μ. For $c>0$, $p>0$, define

$$f_0(x)=f(x),\ f_{k+1}(x)=\frac{d}{dx}\left\{\frac{f_k(x)-f_k((c+k)/p)}{x-(c+k)/p}\right\}$$
$$(k=0, 1, \cdots, N-2).$$

It is shown that the expression

$$(1)\quad p^{-c}\Gamma(c)\sum_{k=0}^{N-1}c(c+1)\cdots(c+k-1)p^{-2k}f_k\left(\frac{c+k}{p}\right)$$

differs for large positive p from

$$(2)\quad \int_0^\infty e^{-px}x^{c-1}f(x)dx$$

only by $O(p^{-2N-c})$. This compares favorably with the classical expansion obtained by Watson's Lemma [A treatise on the theory of Bessel functions, 2nd ed., Cambridge, 1944, p. 236; MR **6**, 64], where the error after N terms is $O(p^{-N-c})$. Also, under additional conditions on $f(x)$, (1) converges for $N\to\infty$. On the debit side, the lack of a convenient generating function for the functions $f_k(x)$ must be noted. Even in the three examples considered in the paper ($c=1$, $f(x)=(1+x)^{-1}$, $(1+x)^{-1}$; $c=\frac{1}{2}$, $f(x)=(2+ix)^{-\frac{1}{2}}$) the expansion is not pushed beyond the first term. Numerical values thus obtained are shown to agree with the true values to two significant digits, even for moderate values of p such as $p=4$. *P. Henrici.*

See also: Functions of Real Variables: Arnol'd. **Numerical Methods:** Kantorovitch.

Trigonometric Series and Integrals

Izumi, Shin-ichi; Satô, Masako; and Sunouchi, Genichirô. Fourier series. XIV. Order of approximation of partial sums and Cesàro means. Proc. Japan Acad. **33** (1957), 114–118.

Let f be a Lebesgue integrable function and let $S_n(x)$ and $\sigma_n^\alpha(x)$ be the nth partial sums and the nth Cesàro means of the Fourier series of f at x. Also let $\theta(u)=u\{f(x+u)+f(x-u)-2f(x)\}$. Various sufficient conditions are given in order that (*) $S_n(x)-f(x)=O(n^{-\alpha})$ or (**) $\sigma_n^\alpha(x)-f(x)=O(n^{-\alpha})$. For example, if $f\in \mathrm{Lip}\ \alpha$, $0<\alpha<1$ and $\int_0^t|d\theta(u)|=O(t^{1+\alpha})$ then (*) holds, while if $\theta(u)$ is of bounded variation on $0\leq u\leq\delta$, $\delta>0$ and $\int_0^t|d\theta(u)|=O(t^{1+\alpha})$, $0<\alpha<1$, then (**) holds. *P. Civin.*

Kanno, Kôsi. On the Cesàro summability of Fourier series. III. Tôhoku Math. J. (2) **9** (1957), 27–36.

[For parts I, II see same J. (2) **7** (1955), 110–118, 265–278; MR **17**, 361, 964]. Let $\phi_x(t)$ be the αth integral of the even integrable function $\phi(t)$ of period 2π. Let $\phi(t)\sim\sum_{n=1}^\infty a_n\cos nt$ and let S_n^β be the βth Cesaro sum of the Fourier series for $\phi(t)$ at $t=0$. The following theorem of Tauberian type, which is related to the summability theorem of part II quoted above, is obtained. If $a_n>-K(\log n)^\alpha/n$ ($\alpha>0$) as $n\to\infty$, with K constant, and if $S_n^\beta=o(n^\beta/(\log n)^\gamma)$ ($\beta, \gamma>0$) as $n\to\infty$, then $\phi_\mu(t)=o(t^\mu)$ as $t\to0$ for $\mu=\alpha(1+\beta)/(\alpha+\gamma)$. *P. Civin.*

Izumi, Shin-ichi. Fourier series. V. A divergence theorem. Proc. Japan Acad. **33** (1957), 1–3.

The author claims that a result due to A. G. Dzwarsejsvili [Soobšč. Akad. Nauk Gruzin. SSR **11** (1950), 403–407; MR **14**, 635] is incorrect, and gives a correct version. *K. Chandrasekharan* (Bombay).

Satô, Masako. Fourier Series. VI. A convergence theorem. Proc. Japan Acad. **33** (1957), 4–9.

If the Fourier series of a function $f(t)$ is summable $(C, 1)$ at a point, then a suitable Tauberian condition imposed on the Fourier coefficients would imply convergence of the series at that point; here the author imposes instead suitable conditions on f itself to deduce convergence. *K. Chandrasekharan* (Bombay).

Izumi, Shin-ichi; and Satô, Masako. Fourier series. XVI. The Gibbs phenomenon of partial sums and Cesàro means of Fourier series. I, II. Proc. Japan Acad. **33** (1957), 284–288, 289–292.

The paper is mainly concerned with the Gibbs phenomenon at points of discontinuity of the second kind. Thus the condition $\int_0^h(f(x+u)-f(x-u))du=o(h/(\log 1/h))$, as $h\to0$ uniformly in x, prevents the Gibbs phenomenon., Similarly, the condition $\int_0^h(f(x+u)-f(x-u))du=o(h)$, uniformly in x, prevents the Gibbs phenomenon for any

Cesàro mean of positive order. [Cf. Izumi and Satô, Kôdai Math. Sem. Rep. 8 (1956), 164–180; MR 19, 138.]

W. W. Rogosinski (Boulder, Colo.).

Satô, Masako. Fourier series. XVII. Order of partial sums and convergence theorem. Proc. Japan Acad. 33 (1957), 298–303.

Let $f(x) \in L$ on $\langle 0, 2\pi\rangle$ and let $s_n(x)$ be the nth partial sum of its Fourier series. Conditions for $s_n(x)=o(\log n)^\alpha$ and $s_n(x)=o(\log\log n)^\alpha$, $0\leqq\alpha\leqq 1$, are given. In the first case, e.g., $t^{-1}\int_0^t (t-u)(f(x+u)+f(x-u))du=o[t(\log 1/t)^\alpha]$ plus $t^{-1}\int_0^t(t-u)(f(\xi+u)-f(\xi-u))du=o[t/(\log 1/t)^{1-\alpha}]$, as $t\to 0$, uniformly in ξ near x, are stipulated. Under the same conditions,

$$\frac{a_0}{2}+\sum_{n=1}^{\infty}\frac{a_n\cos nx+b_n\sin nx}{1+\varepsilon(\log n)^\alpha}\to f(x),$$

as $\varepsilon\to 0$, in the Lebesgue set. [Cf. Satô, same Proc. 32 (1956), 529–534; MR 18, 735.] *W. W. Rogosinski.*

Bhatt, S. N. On the negative order summability of a Fourier series at a point. Proc. Nat. Inst. Sci. India. Part. A. 23 (1957), 306–311.

$f(t)$ is summable over $(-\pi, \pi)$ with Fourier series $\frac{1}{2}a_0+\sum(a_n\cos nt+b_n\sin nt)$. Write

$$\phi(t)=\tfrac{1}{2}\{f(x+t)+f(x-t)-2s\}$$

$$\phi_\alpha(t)=\alpha t^{-\alpha}\int_0^t (t-u)^{\alpha-1}\phi(u)du,\ \alpha>0.$$

Using a criterion of Bosanquet and Offord [Proc. London Math. Soc. (2) 40 (1935), 273–280], the author extends a result of Bosanquet to prove that if $0<\alpha<1$ and (i) $\phi_\alpha(t)$ is of bounded variation in an interval $(0, y)$, (ii) $\phi_\alpha(t)\to 0$ at $t\to 0$, (iii) $a_n\cos nx+b_n\sin nx=o(n^{\alpha-1})$, then the Fourier series of $f(t)$ is summable by Cesàro means of order $\alpha-1$ at the point x. *H. G. Eggleston.*

Ul'yanov, P. L. On divergence of Fourier series. Uspehi Mat. Nauk (N.S.) 12 (1957), no. 3(75), 75–132. (Russian)

The author surveys the literature on sets of divergence of trigonometric series and of Fourier series. He gives detailed proofs of various important results that culminate in Zeller's theorem: Each set of type G_δ on $[0, 2\pi)$ is the set of divergence of a Fourier series [Arch. Math. 6 (1955), 335–340; MR 16, 1015]. And he concludes with a list of unsolved problems that center around the question whether every set of type $G_{\delta\sigma}$ is the set of divergence of a trigonometric series. *G. Piranian* (Ann Arbor, Mich.).

Shapiro, Victor L. Uniqueness of multiple trigonometric series. Ann. of Math. (2) 66 (1957), 467–480.

In n-dimensional Euclidean vector space E_n, if $x=\{x_i\}$, $y=\{y_i\}$, the scalar product is $(x, y)=\sum x_iy_i$ and the norm $|x|=(x, x)^{\frac{1}{2}}$. T_n denotes the 'torus' $[-\pi<x_i\leqq\pi]$, and $m=\{m_i\}$ denotes an integral lattice point. Let $T(x)=\sum_m a_m\exp[i(m, x)]$ be a multiple trigonometrical series. It is assumed throughout that (*) $\sum_{R-1<|m|\leqq R}|a_m|=o(R)$ as $R\to\infty$. The lim sup and lim inf, as $t\to+0$, of the Abel sums $\sum a_m\exp[i(m, x)-|m|t]$ are denoted by $f^*(x)$ and $f_*(x)$, respectively. The main results are: Th. 1. If (i) $\bar a_m=a_{-m}$; (ii) f^* and f_* are finite on T_n; and (iii) $f_*(x)\geqq A(x)$ where $A(x)\in L$ on T_n, then $f_*\in L$ on T_n and T is its Fourier series. Th. 2. Let $n\geqq 2$ and $q\in T_n$. If (i) f^* and f_* are finite on T_n-q; and (ii) f^* and $f_*\in L$ on T_n, then T is the Fourier series of f_*.

Theorem 1 generalises Verblunsky's familar result for $n=1$, while Theorem 2 is false for $n>1$. The initial assumption (*) is also essential. *W. W. Rogosinski.*

See also: Special Functions: Shapiro. **Sequences, Series, Summability:** Hirokawa. **Approximations, Orthogonal Functions:** Morozov.

Integral Transforms

Rooney, P. G. A property of the Laplace transformation. Proc. Amer. Math. Soc. 8 (1957), 883–886.

It is known that if

$$f(s)=\mathscr{L}(\phi(t); s)=\int_0^\infty e^{-st}\phi(t)dt,$$

then under certain conditions

$$\mathscr{L}(\phi(t^2); s)=\tfrac{1}{4}\pi^{-\frac{1}{2}}s\int_0^\infty e^{-sy}y^{-3/2}f(s^2/4y)dy.$$

The author generalises this result by obtaining the following theorem. If either (a) $\nu\geqq 0$, and $e^{-\gamma t^2}\phi(t)\in L(0,\infty)$ for some $\gamma>0$, or (b) $\nu<0$, and $t^\nu e^{-\gamma t^2}\phi(t)\in L(0, \infty)$ for some $\gamma>0$, then $\mathscr{L}(\phi(t); s)$ exists for all $s>0$,

$$\int_0^\infty t^{\nu+1}K_\nu(st)\phi(t^2)dt$$

exists for all $s>\gamma$, and for all $s>\gamma$,

$$\int_0^\infty t^{\nu+1}K_\nu(st)\phi(t^2)dt=2^{\nu-2}s^{-\nu}\int_0^\infty e^{-sy}y^{\nu-1}f(s^2/4y)dy.$$

Here $K_\nu(x)$ is the modified Bessel function of the second kind.

{It is to be observed that since $K_\nu(x)\neq O(x^{-\nu})$ as $x\to 0+$ for $\nu=0$, the proof on p. 884 fails for $\nu=0$.}

J. L. Griffith (Kensington).

Jaeckel, K. Integraltransformationen mit Differenzkern, bei denen Kern-, Objekt- und Bildfunktion zum gleichen Typus gehören. Z. Angew. Math. Mech. 37 (1957), 401–403.

Writing the Gauss transform as

$$Y(x)=\pi^{-\frac{1}{2}}\int_{-\infty}^{\infty}H\exp\{-H^2(x-\xi)^2\}y(\xi)d\xi,$$

and putting $y(\xi)=\pi^{-\frac{1}{2}}h_1\exp\{-h_1^2(\xi-\alpha)^2\}$, we find that $Y(x)=\pi^{-\frac{1}{2}}h_2\exp\{-h_2^2(x-\alpha)^2\}$, where $h_2^{-2}=H^{-2}+h_1^{-2}$, which gives a special case of the formula

(i) $$h_2K(h_2(x-\alpha))=\int_{-\infty}^{\infty}HK(H(x-\xi)(\cdot h_1K(h_1(\xi-\alpha))d\xi.$$

The aim of this paper is to find non-negative functions $K(x)$ which are integrable square in the interval $(-\infty, \infty)$, with $\int_{-\infty}^{\infty}K(x)dx=1$, and which satisfy (i) with H, h_1 and h_2 positive parameters.

The author adds some additional restrictions on $K(x)$ and on the type of equation connecting H, h_1 and h_2. His most general result is

(ii) $$K(x)=\frac{1}{2\pi}\int_{-\infty}^{\infty}\exp\{-a|v|^c\}\exp\{-ixv\}dv,$$

where a is a positive constant to be determined.

{It appears to the reviewer that the derivation of (ii) restricts c to positive values $\geqq 2$.} *J. L. Griffiths.*

See also: **Lie Groups and Algebras:** Berezin. **Numerical Methods:** Huss and Donegan. **Probability:** Spitzer. **Optics, Electromagnetic Theory, Circuits:** Delavault.

Ordinary Differential Equations

★**Hurewicz, Witold. Lectures on ordinary differential equations.** The Technology Press of the Massachusetts Institute of Technology, Cambridge, Mass.; John Wiley & Sons, Inc., New York, 1958. xvii+122 pp. $5.00.

This excellent little book is essentially a reprint of notes by J. P. Brown from lectures given in 1943 by the late author at Brown University, to which have been added a preface by N. Levinson, the obituary "Witold Hurewicz, in memoriam" by S. Lefschetz [Bull. Amer. Math. Soc. **63** (1957), 77–82], and a short list of relevant books. It covers mainly existence theorems for first order scalar and vector equations, basic properties for linear vector equations, and two-dimensional non-linear autonomous systems. These topics are presented here in highly attractive form with particular emphasis throughout on geometric methods.

Chapter 1 deals with the scalar equation $y'=f(x, y)$. The fundamental uniqueness and existence theorems are proved by means of ε-approximate solutions and the Cauchy-Euler method. Theorems on the dependence of a solution on its initial values and on the continuation of a solution follow. The chapter closes with a discussion of Picard's method. In chapter 2 these results are transferred to the vector equation $\dot{x}=f(x, t)$, and the general theorem on continuity and differentiability of a solution depending on parameters is given. Chapter 3 is concerned with the basic properties of linear vector equations of first order and linear scalar equations of higher order. Reduction of order, Wronskian determinant, Green's function are discussed for the general case. Equations with constant coefficients are treated in detail. Chapter 4 is devoted to a careful discussion of the configurations near the origin of the characteristics of the linear and non-linear autonomous systems: $\dot{x}=P(x, y)$; $\dot{y}=Q(x, y)$; $P(0, 0)=Q(0, 0)=0$. Here, the author's thoroughly geometric treatment, particularly of the various non-linear cases, is especially attractive. The final Chapter 5 is mainly devoted to a proof of the Poincaré-Bendixson theorem on the existence of limit cycles. A brief discussion of orbital stability and the index of simple singularities rounds out the chapter. *H. A. Antosiewicz.*

Borůvka, O. Über eine Verallgemeinerung der Eindeutigkeitssätze für Integrale der Differentialgleichung $y'=f(x, y)$. Acta Fac. Rerum Nat. Univ. Comenian. Math. **1** (1956), 155–167. (Czech and Russian summaries)

The author proves a uniqueness theorem for the ordinary differential equation $y'=f(x, y)$. The statement of the theorem is too lengthy to be reproduced here in full. However, loosely speaking, it goes as follows: Suppose we can construct continuous functions $\varphi(x; u, v)$ and $\Phi(x; u, v)$ satisfying a number of conditions, the most important of which are (1) $\varphi(x; u, v)=0$ if $u=v$, >0 otherwise, (2) $\lim_{x\to\xi}\varphi(x; u(x), v(x))=0$ for arbitrary solutions $u(x)$, $v(x)$ of $y'=f(x, y)$ passing through (ξ, η), (3) $\varphi_x(x, u, v)+\varphi_u(x, u, v)f(x, u)+\varphi_v(x, u, v)f(x, v)\leq\Phi(x; u, \varphi(x, u, v))$. (4) For each solution $u(x)$ of $y'=f(x, y)$ passing through (ξ, η), $z(x)\equiv 0$ is the only solution to $z'=\Phi(x, u(x), z)$ near $x=\xi$; then uniqueness holds for $y'=f(x, y)$ near (ξ, η). The theorem includes as special cases criteria of Peano, Tonnelli, Bompiani, Osgood and Tamarkin, and Lipschitz. *W. Littman.*

Miachin, V. F. On the system of two Briot and Bouquet equations. Dokl. Akad. Nauk SSSR (N.S.), **114** (1957), 479–482. (Russian)

The system under consideration is the complex domain system

(1) $x(dy_s/dx)=p_{s1}y_1+p_{s2}y_2+F_s(y_1, y_2, x)$, $(s=1, 2)$,

where the P_{sj} are constants and the F_s are holomorphic about x, y_1, $y_2=0$ which vanish for $y_1=y_2=0$. Picard [Traité d'Analyse vol. 3 Ch. 2] showed that if the characteristic roots λ_1, λ_2 of $\|p_{sj}\|$ are not positive integers there is always a holomorphic solution such that

(2) $$y_s\to 0 \text{ as } x\to 0.$$

Also, if $\lambda_1\neq\lambda_2$ and there is no relation $m+m_1\lambda_1+m_2\lambda_2=\lambda_s$ $(s=1, 2)$ (m, m_j non-negative integers whose sum ≥ 2), then there is a whole family of solutions satisfying (2). The author deals with all the exceptional cases and gives (without proofs) expansions for each satisfying (2) in terms of two or three of the quantities x, x^{λ_1}, x^{λ_2}, $x^{\lambda_1}\log x$, $x^{\lambda_2}\log x$. *S. Lefschetz* (Mexico, D.F.).

Wintner, Aurel. On disconjugate linear differential equations. Arch. Math. **8** (1957), 290–293.

According to a result of Lyapunov [cf., e.g., Bieberbach, Einführung in die Theorie der Differentialgleichungen im reellen Gebiet, Berlin, 1956; p. 230; MR **19**, 139] the equation $\ddot{x}+f(t)x=0$ is disconjugate on $[0, 1]$ if $f(t)$ is continuous and non-negative on $[0, 1]$, and if $\int_0^1 f(t)dt\leq 4$, the constant 4 being best possible. The author proves the same assertion under the assumptions that $f(t)$ be continuous on $[0, 1]$ and that $F(t)=\int_t^1 f(s)ds$ satisfy $0\leq F(t)\leq 4$ on $[0, 1]$. He also proves the following comparison theorem. If $f_1(t)$, $f_2(t)$ are continuous on $(0, 1)$ and $\varphi_1(t)=\int_{+0}^t sf_1(s)ds$ exist and satisfy $0\leq\varphi_1(t)\leq\varphi_2(t)$ on $(0, 1)$, then disconjugacy of $\ddot{x}+f_2(t)x=0$ on $(0, 1)$ implies that of $\ddot{x}+f_1(t)x=0$ on $(0, 1)$. *H. A. Antosiewicz.*

Wintner, Aurel. On a principle of reciprocity between high- and low-frequency problems concerning linear differential equations of second order. Quart. Appl. Math. **15** (1957), 314–317.

For the differential equation (*) $d^2x/dt^2+xf(t)=0$, in which $f(t)>0$ for $t>0$, there are various theorems on the asymptotic behaviour of solutions for $t\sim\infty$. The author remarks, as an empirical fact, that there is a correspondence between theorems that concern (a) the case $f(t)$ large and (b) the case $f(t)$ small, for $t\sim\infty$; and he shows that the underlying fact is that (*) is converted into $d^2y/ds^2+y/f(t)=0$ by the substitution

$$s=\int f(t)dt,\quad \frac{dy}{ds}\,\frac{dx}{dt}=-xy,$$

which is significant in the asymptotic context provided $s\to\infty$ as $t\to\infty$. (The substitution is in fact a disguised form of $y=dx/dt$.) *T. M. Cherry* (Melbourne).

Wintner, Aurel. Comments on "flat" oscillations of low frequency. Duke Math. J. **24** (1957), 365–366.

The question raised is under what conditions every solution of $x''+f(t)x=0$ admits the bound $x(t)=O(t^t)$, as $t\to\infty$, with the possible additional requirements being that $x(t)\neq O(t^{t-\varepsilon})$ for any $\varepsilon>0$, or that $x'(t)=O(t^{-\frac{1}{2}})$, or

that $x'(t)=O(t^{-1+\epsilon})$. The requirement $\int^{\infty} t|f(t)-g(t)|dt<\infty$ is given as the ensuring condition that $x''+g(t)x=0$ belongs to such a class if $x''+f(t)x=0$ does; a less complete result for the equivalent stability problem was given by the author in Quart. Appl. Math. 13 (1955), 192–195 [MR 16, 1111], where the relevant result is not cited. This leads to the breaking up of $f(t)$ into positive and negative parts, in the manner of M. L. Boas, R. P. Boas and N. Levinson [Duke Math. J. 9 (1942), 847–853; MR 4, 158]. The author suggests that the case of positive $f(t)$ be solved first. *F. V. Atkinson* (Canberra).

Magnus, Wilhelm; and Shenitzer, Abe. Hill's equation. Part I: General Theory. Div. Electromag. Res., Inst. Math. Sci., New York Univ., Res. Rep. No. **BR-22** (1957), i+42 pp.

The purpose of this report is to give a detailed account of the elements of the theory of Hill's equation, including recent results, and to provide a guide to the extensive literature on special cases. The discussion starts with Floquet's theorem, passes on to the characteristic values, the discriminant and the theorems on stability, on convergence of Hill's type determinants, gives the asymptotic behavior of the characteristic values, and goes into the coexistence problem. Some literature of the years during and after World War II is completely omitted, which is regrettable from several points of view. The authors' aim is thus hardly achieved. *M. J. O. Strutt* (Zurich).

Kondrat'ev, V. A. Elementary derivation of a necessary and sufficient condition for non-oscillation of the solutions of a linear differential equation of second order. Uspehi Mat. Nauk (N.S.) **12** (1957), no. 3(75), 159–160. (Russian)

Let $L(u)$ indicate the second order linear differential operator $u''+pu'+qu$. The author shows by means of a simple and ingenious argument, based upon the change of variable $u=vw$, the known result that a sufficient condition for the non-oscillatory behavior of all solutions of $L(u)=0$ is that there exist one function v which satisfies the conditions $v>0$, $L(v)<0$ for all large t.

R. Bellman (Santa Monica, Calif.).

Mitrinovitch, D. S. Compléments au traité de Kamke. V. Univ. Beogradu. Publ. Elektrotehn. Fak. Ser. Mat. Fiz. no. **11** (1957), 10 pp. (Serbo-Croatian summary)

The author notes that the equation

$$y^{(n+k)}-f(x)L_n(y)=0,$$

where $L_n(y)=x^nD^n(y/x)$, $D=d/dx$, can be reduced to one of order k by the substitution $z=L_n(y)$. He notes that many higher order equations can be reduced to familiar second order ones by this means.

He also notes that the equation $\sum_{k=1}^{n} f_k(x)L_{N_k}(y)=0$ can be reduced by the substitution $y=xz$ to one whose order is the maximum of the N_k's minus the minimum of the N_k's. *E. Pinney* (Berkeley, Calif.).

Koval, P. I. On the asymptotic behaviour of solutions of linear difference equations and linear differential equations. Dokl. Akad. Nauk SSSR (N.S.) **114** (1957), 949–952. (Russian)

An important tool in the study of asymptotic behavior of solutions of linear difference (differential) equations is reduction to l-diagonal (L-diagonal) form. The method was initiated by Cesari [Ann. Scuola Norm. Sup. Pisa (2) **9** (1940), 163–186; Atti Accad. Italia. Mem. Cl. Sci. Fis. Mat. Nat. **11** (1940), 633–695; MR 3, 41; 8, 208] for questions of stability and used by Levinson [Duke Math. J. **15** (1948), 111–126; MR 9, 509] and Rapoport [On some asymptotic methods in the theory of differential equations, Izdat. Akad. Nauk Ukrain. SSR, Kiev, 1954; MR 17, 734]. The author extends the method to include a wider class of equations than those studied by Rapoport and Levinson. Several theorems are given which state necessary and sufficient conditions for difference (differential) systems to be reducible to l-diagonal (L-diagonal) form. Application is made to second order equations with variable coefficients including the case of multiple characteristic roots. *N. D. Kazarinoff.*

Demidovič, B. P. On boundedness of monotonic solutions of a system of linear differential equations. Uspehi Mat. Nauk (N.S.) **12** (1957), no. 2(74), 143–146. (Russian)

The author proves that if the n-vector equation $\dot{x}=P(t)x$ with $P(t)=(p_{ij}(t))$ real, continuous on $[0, \infty)$ and $\int^{\infty} p_{ij}(t)dt<\infty$ $(1\leq i,j\leq n)$ has a monotone solution $x(t)$ on $[t_0, \infty)$, then $x(t)\to x(\infty)<\infty$ as $t\to\infty$.

H. A. Antosiewicz (Providence, R.I.).

Razumichin, B. S. On stability of automatic control systems possessing one control unit. Avtomat. i Telemeh. **17** (1956), 958–968. (Russian)

The author proposes a method for studying the stability of a control system with a nonlinear servomotor and states that the determination of the restrictions on the characteristic of the servomotor to assure stability is reduced to finding the roots of a quadratic equation. It is not quite this simple, but the method is difficult to describe briefly and, if other conditions are satisfied, the problem is eventually reduced to that of finding the roots of a quadratic equation. The differential equation of the control system is of the form

$$\dot{y}_i=\sum_{j=1}^{n} b_{ij}y_j+k_if(s),\ i=1, 2, \cdots, n,\ s=\sum_{i=1}^{n} c_iy_i;$$

f is the characteristic of the servomotor. *J. P. LaSalle.*

Yakubovič, V. A. Extension of some results of Lyapunov to linear canonical systems with periodic coefficients. Prikl. Mat. Meh. **21** (1957), 491–502. (Russian)

Consider the $2k$-dimensional system $dx/dt=\lambda JH(t)x$, where $J=\begin{pmatrix} 0 & I_k \\ -I_k & 0 \end{pmatrix}$, I_k being the kth order unit matrix, and $H(t)$ is piecewise continuous, real and of period ω. Let $X(t, \lambda)$ be the solution of $dX/dt=\lambda JH(t)X$ such that $X(0, \lambda)=I_{2k}$, and suppose that the characteristic equation $\det[X(\omega, \lambda)-I_{2k}]=0$ has no roots $\rho=\pm 1$. Then the real roots ρ fall into positive pairs and negative pairs; the author distinguishes four cases according as the numbers of such positive or negative pairs are even or odd. He establishes necessary and sufficient conditions for the signs of partial sums in the expansions $\det[X(\omega, \lambda)\mp I_{2k}]=$ const. $\lambda^{2m}(1-A_1^{\pm}\lambda^2+A_2^{\pm}\lambda^4-\cdots)$ in these cases. Certain additional restrictions on $H(t)$ imply that only even powers of λ occur; in the case $k=1$ this is very similar to the methods of A. M. Lyapunov [Zap. Imp. Akad. Nauk. Fis.-Mat. Otd. (8) **13** (1902), no. 2]. A procedure for obtaining the $A_r^{\pm}$ is also given. Another result gives similar conditions for $H(t)$ to belong to the central stability zone 0_0^+, which includes all sufficiently small $H(t)>0$ for which the solutions of $dx/dt=JH(t)x$ are bounded.

F. V. Atkinson (Canberra).

Vinograd, R. E. Estimate of the jump of the higher characteristic exponent in the case of small perturbations. Dokl. Akad. Nauk SSSR (N.S.) **114** (1957), 459–461. (Russian)

Consider the associated n-vector systems (1) $\dot{x}=A(t)x$, (2) $\dot{x}=A(t)x+f(t,x)$, where A is an $n\times n$ matrix which is continuous or piecewise continuous on $[0,-\infty)$ and also bounded: $|A|\leqq M$. Furthermore, $f(t,0)\equiv 0$, $t>0$, and f is continuous with $|f(t,x)|<\delta|x|$, where δ is constant. Let L_δ denote the class of all f whose constant $\leqq\delta$. Let also λ_0, λ_f be the largest characteristic (Liapunov) numbers of (1) and (2). While $\lambda_f\geqq\lambda_0$, if $|f(t,x)|$ $\Lambda=\sup\lambda_f$ as $\delta\to 0$ and $f\in L_\delta$, then it may well happen that $\Lambda>\lambda_0$ [See Perron, Math. Z. **31** (1930), 748–766; Vinograd, Prikl. Mat. Meh. **17** (1953), 645–650; MR **15**, 624; Persidskii, Izv. Akad. Nauk Kazah. SSR. Ser. Mat. Meh. **1947**, no. 1; Bylov, dissertation Univ. of Moscow (1954)]. The author gives an upper bound for Λ which, in a certain sense, cannot be improved. Let $X(t)$ be a fundamental matrix of solutions of (1). Let $F(t)$ be a bounded function such that for $\tau<t$ and all t: $|X(t)X^{-1}(\tau)|<C\exp(\int_\tau^t F(\xi)d\xi)$, where C depends solely upon F but not upon τ, t.

Then

$$\Lambda\leqq\Omega=\inf_{0<F<\infty}\limsup_{t\to+\infty}t^{-1}\int_0^t F(\xi)d\xi.$$

S. Lefschetz (Mexico, D.F.).

Vinograd, R. E. The inadequacy of the method of characteristic exponents when applied to non-linear equations. Dokl. Akad. Nauk SSSR (N.S.) **114** (1957), 239–240. (Russian)

Consider $\dot{x}=F(t,x)$ where F is everywhere continuous and lipschitzian in x and $F(t,0)\equiv 0$, and let Λ be the supremum of the Perron order numbers of the solutions $x(t)\not\equiv 0$. It is known that in the linear case, $F(t,x)\equiv A(t)x$, (1) for any $\varepsilon>0$ there is a $B_\varepsilon>0$ such that $|x(t)|\leqq|x(0)|B_\varepsilon\exp(\Lambda+\varepsilon)t$ for $t\geqq 0$, (2) $\Lambda<0$ implies the asymptotic stability of $x=0$; and that in the non-linear case (1) holds with B_ε also dependent upon $x(0)$, (2) is false [Vinograd, Mat. Sb. N.S. **41(83)** (1957), 431–438; MR **19**, 416]. The author now states that (2) holds in the non-linear case provided B_ε exists for each sphere $|x(0)|=v(<R)$ and $B_\varepsilon\to\infty$ with $1/r$. He gives two examples, one of which is the example discussed in the paper referred to above [cf., also, the review cited above].

H. A. Antosiewicz.

Bailey, H. R.; and Gambill, R. A. On stability of periodic solutions of weakly nonlinear differential systems. J. Math. Mech. **6** (1957), 655–668.

The 'weakly non-linear systems' of the paper belong to the context of non-linear oscillations: the equations involve a small parameter ε, for $\varepsilon=0$ they are trivially soluble and represent a set of uncoupled harmonic oscillations, while for $\varepsilon\neq 0$ they correspond to forced oscillations. The particular concern is to calculate the characteristic exponents, and thence decide the stability, of a subharmonic solution. This is done by adaptation of a method of L. Cesari, R. A. Gambill and J. K. Hale for investigating periodic solutions of weakly non-linear systems [Cesari and Hale, Proc. Amer. Math. Soc. **8** (1957), 757–764, and references there cited; MR **19**, 142]. Explicit criteria for stability are found for two systems of van der Pol type, of orders 2 and 4 respectively.

T. M. Cherry (Melbourne).

Lykova, O. B. On the behaviour of solutions of a system of differential equations in the vicinity of a closed orbit. Ukrain. Mat. Ž. 9 (1957), 419–431. (Russian. English summary)

By method and content this paper is closely related to the monograph of Bogoliubov and Mitropolskii, "Asymptotic methods in the theory of non-linear oscillations" [Gostehizdat, Moscow, 1955; MR **17**, 368]. Consider the two related systems (1) $\dot{x}=X(x)+\varepsilon X^*(t,x,\varepsilon)$; (2) $\dot{x}=X(x)$ where x, X, X^* are n-vectors and ε is positive and small. Suppose that the following conditions hold: (a) (2) has a known solution (3) $x=x^0(\omega t+\varphi)$, $\psi=\omega t+\varphi$ with period 2π in ψ, depending on two arbitrary parameters a, φ, where in general ω depends upon a; (b) the variation equation of (2) relative to (3) then has two zero characteristic roots, and its remaining roots have negative real parts; (c) $\Delta(a)=0$ for $a\in A$ where

$$\Delta(a)=\min\left|\frac{\partial x^0}{\partial\psi},\frac{\partial x^0}{\partial a},\ \tfrac{1}{2}(A+\bar{A})\right|$$

(A described below); (d) in a certain neighborhood U of (3) (for $\varepsilon=0$) in the space (x,ε), the functions $X+\varepsilon X^*$ and their partials with respect to the x_i are uniformly continuous and bounded for all t; (e) X^* has in U the period 2π in t.

The basic process rests upon a change of variables from x to a, ψ and $h=(h,\cdots,h_n)$, of the form $x=x^0(\psi,a)+\frac{1}{2}\{A(\psi,a)+\overline{Ah}\}$ where A has the period 2π in ψ with bounded uniformly continuous partials of all orders with respect to ψ, a, and this is the A of condition (c).

Under these restrictions the author shows that (1) has in U a two-parameter manifold M of solutions, and that all other solutions which start in a certain region tend to M.

S. Lefschetz (Mexico, D.F.).

Zubov, V. I. Investigation of the problem of stability for systems of equations with homogeneous right-hand members. Dokl. Akad. Nauk SSSR (N.S.) **114** (1957), 942–944. (Russian)

Let $\mu=p/q$ be a positive rational fraction with q odd. A continuous function of fixed sign of the n-vector x, $X(x)$, defined for all x, is said to be homogeneous of degree μ whenever $X(cx)=c^\mu X(x)$, whatever the real constant c, where one considers c positive for p even and of the sign of c for p odd. A similar function $V(x)$ is said to be homogeneous positive of degree $m>0$ (denoted by $V^{[m]}$) whenever $V(cx)=|c|^m V(x)$.

Consider now the system (1) $\dot{x}=X^{(\mu)}(x)$. If $X=X(t,X^0)$, $X(0,X^0)=X^0$ is a solution, then $cX(c^{\mu-1}t,X^0)=X(t,cX^0)$ is also. A certain number of results announced without proofs are as follows.

Th. 1. (a) For p even, the origin cannot be asymptotically stable (a.s.) for (1); (b) for p odd and $\mu\neq 1$, the origin can be a.s. for (1) but only in the real domain; (c) for $\mu=1$ the origin can be a.s. in the complex domain; also (2) $\|X(t,X^0)\|<At^{-\alpha}$ if $\|X^0\|=1$, with A, $\alpha>0$, if the origin is a.s. and if $X^{(\mu)}$ is of class C^ν, $\nu\geqq 1$.

Th. 2. If (2) holds, there is an $m>\mu-1$ and $W^{[m]}$, $V^{[m+1-\mu]}$, with V and $-W$ definite negative and $\dot{V}=W$. If $X^{(\mu)}$ is of class C^ν, $\nu>1$, and one may choose W such that V is of class C^ν, then it is the only solution of

$$\sum\frac{\partial V}{\partial x_i}X_i^{(\mu)}=W^{[m]},\quad\sum x_i\frac{\partial V}{\partial x_i}=(m+1-\mu)V$$

such that $V(0)=0$.

Consider now the system (3) $\dot{x}=X^{(\mu)}+f(x,t)$, and let it

have a solution $x(t, X^0, t_0)$ for all X^0, t_0. Th. 3. Let $X^{(\omega)}$ be of class C^ν and let $\|f\|<c\|X\|^\lambda$ for $\|x\|$ large enough and $c>0$ and small when $\lambda=\mu$. A necessary and sufficient condition that, whatever such f, all solutions of (3) be bounded for $t\geqq t_0$ is that (2) hold, and then the origin is stable for (3).

Take now the system (4) $\dot{x}=X(x, y)$; $\dot{y}=Py+Y(x, y)=\Phi(x, y)$, where x is a k-vector, y an n-vector, P a constant matrix whose characteristic roots all have negative real parts and X, Y are holomorphic about $x=0$, $y=0$. Let also $u(x)$ be the (analytic) solution of $\Phi(x, y)=0$. Th. 4. If $X(x; u)=0$, then (4) has k holomorphic solutions and the origin is stable for it.

Let $\bar{X}^{(\omega)}$ be the lowest degree vector-term in x in the expansion of $X(x, u)$ in powers of the x_s. Th. 5. If the origin is a.s. for $x=\bar{X}^{(\omega)}$, then it is also a.s. for (4). An estimate is also given for solutions starting near enough to the origin. *S. Lefschetz* (Mexico, D.F.).

Zadiraka, K. V. On the integral manifold of a system of differential equations containing a small parameter. Dokl. Akad. Nauk SSSR (N.S.) **115** (1957), 646–649. (Russian)

The author considers the system (1) $dx/dt=f(t, x, z, t/\mu)$, $\mu dz/dt=F(t, x, z)$ with initial data $x(t_0)=x^0$, $z(t_0)=z^0$, where x, f are n-vectors and z, F are m-vectors, and compares its solution, as $\mu\to 0^+$, with that of the reduced system (2) $d\bar{x}/dt=f_0(t, \bar{x}, \bar{z})$, $\bar{z}=\varphi(t, \bar{x})$, $\bar{x}(t_0)=x^0$, where $z=\varphi(x, t)$ is a zero of F and $f_0=\lim_{T\to\infty} T^{-1}\int_0^T f(t, x, z, \nu)d\nu$. The assumptions are to the effect that for $-\infty<t<\infty$, $x\in G$, $|z-\varphi(x, t)|\leq\rho$, and $0<\mu<\mu^*$, the functions f, F, and φ and their derivatives are sufficiently regular and satisfy certain Lipschitz conditions and that the roots of the characteristic equation $\det\|pI-F(t, x, 0)\|=0$ all have negative real parts bounded away from zero. Two theorems are proved. The first states that there exists $\mu_0>0$ such that for $0<\mu<\mu_0$, system (1) has a solution of the form $z=\varphi(x, t)+\psi(t, x, \mu)$ in which ψ is defined for $-\infty<t<\infty$ and $x\in G$ and satisfies the inequalities $|\psi|\leq D(\mu)<\rho$, $|\psi(t, x', \mu)-\psi(t, x'', \mu)|\leq\Delta(\mu)|x'-x''|$ where D and $\Delta\to 0$ as $\mu\to 0^+$. The second theorem guarantees that if the solution of (2) has a certain regularity, then so does the solution of (1).

With t/μ deleted, system (1) reduces to one considered by Gradstein [same Dokl. (N.S.) **65** (1949), 789–792; MR **10**, 708] on a finite interval of t. *N. D. Kazarinoff.*

Egorov, V. G. The stability of the solutions of periodic systems of total differential equations. Dokl. Akad. Nauk SSSR (N.S.) **114** (1957), 11–13. (Russian)

The systems considered in this paper are of the form

$$dx=P(u, x)du+Q(v, x)dv, \tag{1}$$

where x, P, Q are n-vectors and u, v independent variables (all real) and where (1) is a total differential. Such systems have already been dealt with by Nemitzkii [Mat. Sb. N.S. **23(65)** (1948), 161–186; MR **10**, 259]; Barbashin [C. R. (Dokl.) Acad. Sci. URSS (N.S.) **55** (1947), 279–282; Mat. Sb. N.S. **27(69)** (1950), 455–470; MR **8**, 589; **12**, 422] and the author [Dokl. Akad. Nauk SSSR (N.S.) **102** (1955), 677–680; MR **17**, 371]. The basic existence theorem goes back to T. Y. Thomas [Ann. of Math. (2) **35** (1934), 730–734]. The author deals mainly with a linear system,

$$dx=p(u)x\,du+q(v)x\,dv, \tag{2}$$

where p, q are matrices of order N. Here the integrability condition becomes: p and q commute. There is also the associated matrix equation

$$dX=p(u)X\,du+q(v)X\,dv. \tag{3}$$

If $\bar{X}$ is a non-singular solution of (3), then every solution is of the form $X=\bar{X}C$, C constant. Hence, if $\bar{X}_u$, $\bar{X}_v$ are normalized solutions of (3) with, respectively, v or u zero then a normalized solution of (3) is $X=\bar{X}_u\bar{X}_v=\bar{X}_v\bar{X}_u$. The system (2) is reducible if a linear transformation $y=L(u, v)x$ yields a system $dy=Ay\,du+By\,du$ with A, B constant (and of course permutable). If $x=0$ is a solution of (2), say, for all u, $v\geqq 0$, and it is stable or asymptotically stable [see Egorov, loc. cit.], then it is the same for (2). A number of results are stated (proofless).

Th. 1. If p and q are periodic then (2) is reducible.

Th. 2. In (1), let $P=p(u)x+P(u, x)$, $Q=q(v)x+Q(v, x)$, p and q periodic and let the systems

$$dx=P(u, x)du,\quad dx=Q(v, x)dv \tag{4}$$

be stable according to the first approximation (Krasovskii). Then the same will hold for

$$\frac{dx}{dt}=\varphi(t)P\Big(\int_0^t\varphi dt, x\Big)+\psi(t)Q\Big(\int_0^t\psi dt, x\Big) \tag{5}$$

provided that p and q commute and that

$$\operatorname{mod}(P, Q)<M\|x\|, \tag{6}$$

where the constant M is small enough [Egorov, loc. cit., Th. 7].

Th. 3. The origin is asymptotically stable for (1) if M in (6) is small enough and the characteristic numbers of Lyapunov of the systems (4) are all positive. If p and q commute this holds also for the system (2). [Egorov loc. cit. Th. 3]. As a consequence, (5) is asymptotically stable at the origin if φ, ψ are continuous, bounded positive functions and at least one of $\int_0^\infty\varphi\,dt$, $\int_0^\infty\psi\,dt$ diverges.

S. Lefschetz (Mexico, D.F.).

Saltykow, M. N. Le théorème de A. M. Liapounoff sur la stabilité des solutions d'équations différentielles. J. Math. Pures Appl. (9) **36** (1957), 229–234.

Démonstration du théorème de Liapounoff utilisant les "facteurs de d'Alembert" introduits par l'A. dans une note antérieure [Acad. Serbe Sci. Publ. Inst. Math. **2** (1948), 190–204; MR **10**, 298]. *H. A. Antosiewicz.*

Taam, Choy-tak. The solutions of nonlinear differential equations. III. Duke Math. J. **24** (1957), 511–519.

[For Part II see Johnson and Taam, J. Math. Mech. **6** (1957), 383–392; MR **19**, 417.] The equation studied is $x''+p(t)x+2q(t)x^3=F(t)$, where the real functions p, q and F have a common period L, p and q are even and have positive lower bounds, F is odd and has a non-negative lower bound; p, q, F are also bounded and Lebesgue-measurable. The results give upper bounds for L, subject to which there exist solutions of period L with specified numbers of zeros in a period, and either in or out of phase with $F(t)$. For example, if $L\leq 2\pi(p_2+3(2F_2^2q_2)^{\frac{1}{2}})^{-\frac{1}{2}}$, where p_2, q_2 and F_2 are least upper bounds of p, q and F, there are two solutions of period L vanishing only at multiples of $\frac{1}{2}L$, such that $x'(0)<0$, and such that x' has just one zero between 0 and $\frac{1}{2}L$. Subharmonics are also considered. The methods resemble those of Part I [J. Math. Mech. **6** (1957), 287–300; MR **19**, 417].

F. V. Atkinson (Canberra).

Faure, Robert. Sur les équations différentielles non linéaires à coefficients périodiques. C. R. Acad. Sci. Paris **244** (1957), 2767–2769.

The author establishes the existence of a periodic solution for a class of non-linear ordinary differential equations of order n with periodic coefficients. *C. J. Titus.*

Faure, Robert. Sur les systèmes d'équations différentielles du premier ordre, non linéaires, à coefficients périodiques. C. R. Acad. Sci. Paris **244** (1957), 3022–3025.

Es wird eine Methode angedeutet, welche ermöglicht die Bedingungen der Existenz einer periodischen Lösung des folgenden Systems von Differentialgleichungen zu gewinnen: $dx_i/dt+\sum_k p_{ik}x_k=f_i(x_k,t)$. Die p_{ik} sind reelle Konstanten und ihre Determinante ist von Null verschieden. Die Funktionen $f_i(x_k,t)$ sind periodisch mit derselben Periode T und von der Form

$$f_i(x_k,t)=a\sum_k b_{ik}x_k+\sum_{pqr} C_{i,pqr}x_1^p x_2^q x_n^r.$$

M. Zlámal (Brno).

Peretiagin, B. M. On the number of limit cycles of equation

$$\frac{dy}{dx}=\frac{cx+dy+P(x,y)}{ax+by+Q(x,y)},$$

where $P(x,y)$ and $Q(x,y)$ are homogeneous polynomials of degree n. Dokl. Akad. Nauk SSSR (N.S.) **114** (1957), 29–32. (Russian)

The basic result of this very obscure note is this. Given

$$\frac{dy}{dx}=\frac{-x-a_1y+P(x,y)}{y-a_1x+Q(x,y)},$$

where P and Q are forms of degree $n-1$. When the coefficients of P, Q vary there appear at most $n+1$ limit-cycles in a certain neighborhood of the origin. In a suitable neighborhood of a point of the domain of coefficients and of the origin there appear $(n+2)/2$ if n is even, $(n+3)/2$ if n is odd, limit-cycles. *S. Lefschetz* (Mexico, D.F.).

Belkina, M. G. Asymptotic representations of spheroidal functions with an azimuth index $m=1$. Dokl. Akad. Nauk SSSR (N.S.) **114** (1957), 1185–1188. (Russian)

The author studies the eigenvalue problem $Y''+c^2p(\eta)Y=0$, $Y(-1)=Y(1)=0$ for large values of the parameter c, where $p(\eta)=1+\beta(1-\eta^2)^{-1}$, β being the eigenvalue. The solutions can be expressed in terms of the spheroidal wave function $S_{1,c}^{(1)}(\eta)$ [in the notation of J. A. Stratton et al., Elliptic cylinder and spheroidal wave functions, Wiley, New York, 1941; MR **4**, 281]. Using the methods of a paper by A. A. Dorodnitsyn [Uspehi Mat. Nauk (N.S.) **7** (1952), no. 6(52), 3–96; MR **14**, 876], taking into account both the pole of $p(\eta)$ and the turning point, the author obtains an asymptotic representation of the solution in terms of confluent hypergeometric functions. From this he deduces an asymptotic formula for the eigenvalues. Numerical values for the first 8 to 13 eigenvalues corresponding to $c=3$, 5, 7 agree to about 3 significant digits with the exact values given by Stratton [loc. cit.]. This agreement is shown to be much better than the one obtained by using Airy functions (taking into account the turning point only) or Bessel functions of order 1 (taking into account the pole of $p(\eta)$ only).

P. Henrici (Los Angeles, Calif.).

Greguš, Michal. Über einige neue Randwertprobleme einer Differentialgleichung dritter Ordnung. Czechoslovak Math. J. **7**(82) (1957), 41–47. (Russian. German summary)

Der Verfasser beschäftigt sich mit den Eigenwertaufgaben bei der Differentialgleichung $y'''+2A(x,\lambda)y'+[A'(x,\lambda)+b(x,\lambda)]y=0$. Er betrachtet fünf Typen von Randbedingungen: (1) $y(a)=y'(a)=y(b)=0$; (2) $y(a)=y'(a)=y'(b)=0$; (3) $y(a)=y(b)=y'(c)=0$; (4) $y(a)=y'(b)=y(c)=0$; (5) $y(a)=y'(b)=y'(c)=0$, wo $a<b<c$. Unter gewissen Bedingungen über die Koeffizienten $A(x,\lambda)$ und $b(x,\lambda)$ beweist er mit Hilfe eines Oszillationssatzes von Sansone [Univ. Nac. Tucumán. Rev. A. **6** (1948), 195–253; MR **10**, 300] die Existenz unendlich vieler reeller Eigenwerte und charakterisiert die zugehörigen Eigenfunktionen durch die Anzahl ihrer Nullstellen im Intervall $\langle a,b\rangle$, bzw. $\langle b,c\rangle$. *M. Zlámal* (Brno).

Bott, Raoul. On the iteration of closed geodesics and the Sturm intersection theory. Comm. Pure Appl. Math. **9** (1956), 171–206.

This paper presents a new approach to the boundary value problem for $Ly-\lambda y=0$, where L is a self-adjoint second order differential operator on the vector variable y in complex Euclidean E^n; by the usual transformation this is rewritten as $u'=A_\lambda u$, with $u(t)\in E^{2n}$. Let $X_\lambda(t)$ denote the fundamental matrix solution of this equation, with $X_\lambda(0)=I$; and let J be the $2n\times 2n$ matrix $\begin{pmatrix}0 & -1\\ 1 & 0\end{pmatrix}$. A Hermitian boundary condition B is a pair (M,N) of matrices such that (a) $Mv=Nv=0$ implies $v=0$, and (b) $M^*JM=N^*JN$. A non-trivial solution $u(t)$ satisfies B on $[0,a]$ if there is a $v\neq 0$ with $u(0)=Mv$, $u(a)=Nv$. One verifies that $X_\lambda(t)$ always lies in the subgroup $\mathscr{H}$ of the general linear group $GL(2n,C)$ defined by $X^*JX=J$. Let $B^0=\{X\in\mathscr{H}: XM-N$ has a non-trivial nullspace$\}$. Solutions $u(t)$ satisfying B on $[0,a]$ correspond then to λ-values with $X_\lambda(a)\in B^0$, and the boundary value problem is thus transformed into the study of the intersections of the curve $\lambda\to X_\lambda(a)$ and the set B^0. The latter set is a submanifold of $\mathscr{H}$ with singularities (the X for which the nullspace of $XM-N$ has dim ≥ 1), and with $\dim B^0=\dim\mathscr{H}-1$. It is shown that B^0 is covered by a locally finite cycle γ_B. The basic result (Th. V) is that, for any admissible interval τ of the λ-axis, the number (with multiplicity) of eigenvalues of the boundary value problem which lie in τ is equal to the topological intersection number of the cycle γ_B and the curve $\lambda\to X_\lambda(a)$, restricted to τ. The proof of this involves a number of subtle topological considerations. To treat the singularities of B^0, the concept of resolution, reminiscent of Hopf's σ-process, is introduced: Let P denote complex projective $(2n-1)$-space (space of lines in E^{2n}). The set $B^{(1)}$ in $\mathscr{H}\times P$, defined by $(X,p)\in B^{(1)}\Leftrightarrow p\in$ nullspace of $XM-N$, is the 1-resolution of B^0. It is an algebraic manifold that projects onto B^0 under the projection of $\mathscr{H}\times P$ onto $\mathscr{H}$. The pre-images of the singular points of B^0 are certain complex projective spaces. $B^{(1)}$ carries a cycle Γ_B (projecting onto γ_B); this needs a detailed orientability consideration. Another concept that plays a role is that of clean intersection of manifolds; this refers roughly to the following situation: U, V, W are manifolds, U, V immersed in W; $\dim U+\dim V=\dim W$; but U and V do not intersect, as they ought to, in isolated points; instead they intersect along (simply connected compact) manifolds such that at each point the intersection of the tangent spaces of U and V is the tangent space to the intersection

manifold. The intersection number of U and V can then be computed in terms of the "self"-intersection numbers of the intersection manifolds.

The basic result leads to the principal application, the description, for a periodic operator L, of the nullity and index function N and Λ (for complex z, with $|z|=1$, the values $N(z)$ and $\Lambda(z)$ are the multiplicity of 0 and the number of negative eigenvalues of the boundary value problem $u(t+1)=z\cdot u(t)$, respectively. It is shown that the Poincaré matrix $X_0(1)$ determines N (trivially) and determines Λ up to an additive constant, in a somewhat complicated fashion, involving the nullity of certain variations of $X_0(1)$. These results are then applied to the study of nullity and index (in the sense of Morse) of a closed geodesic, and yield refinements and extensions of results of Hedlund [Trans. Amer. Math. Soc. **34** (1932), 75–97] and Morse and Pitcher [Proc. Nat. Acad. Sci. U.S.A. **20** (1934), 282–287], relating nullity and index to the eigenvalues of the Poincaré matrix.

The basic result is also used to give new proofs of the comparison and oscillation theorems and the index theorem of Morse. All these facts emerge now as statements about the behavior of the intersection number of two cycles, under deformations of the cycles, etc.

H. Samelson (Ann Arbor, Mich.).

Dickson, Douglas G. Expansions in series of solutions of linear difference-differential and infinite order differential equations with constant coefficients. Mem. Amer. Math. Soc. no. **23** (1957), 72 pp. $1.70.

The equations in question are

$$(1)\qquad \sum_{j=1}^{n}\sum_{k=0}^{m_j} A_{jk}F^{(k)}(z+\omega_j)=0,$$

$$(2)\qquad \sum_{k=0}^{\infty} A_k F^{(k)}(z)=0.$$

These have characteristic equations in ζ, say, which may be obtained formally by substituting $e^{\zeta z}$ for F. A root, ζ, of order $q+1$ of a characteristic equation implies "fundamental" solutions of the type $z^p e^{\zeta z}$, $p=0, 1, \cdots, q$, of the original functional equation. The object of this memoir is to expand either arbitrary functions or solutions of the functional equations in terms of these fundamental solutions. The functional equations are sufficiently general that the expansion problems for all common special functions appear to be included.

The first section is an introduction and summary of results. The second section contains a careful and detailed investigation of the properties of functions which are substantially the exponential sums in the characteristic equations.

In the third section the author introduces some contour integral operators to be used in the subsequent theory. The important fundamental idea here appears to be due to Kitagawa [Jap. J. Math. **13** (1937), 233–332] who avoided the usual rather artificial conditions at infinity on the solutions of difference-differential equations by employing finite contour integral operators. A similar device employing the finite Fourier transform was used by Titchmarsh [J. London Math. Soc. **14** (1939), 118–124].

The fourth section is concerned with the expansion of arbitrary functions in series of fundamental solutions of (1), and with the expansion of solutions of (1) in fundamental solution series in more extended regions. In the fifth section analytic solutions of (2) are expanded in series of fundamental solutions. *E. Pinney.*

Littlewood, J. E. On non-linear differential equations of the second order. IV. The general equation

$$\ddot{y}+kf(y)\dot{y}+g(y)=bkp(\varphi),\quad \varphi=t+\alpha.$$

Acta Math. **98** (1957), 1–110.

Of the earlier papers of this series [I, J. London Math. Soc. **20** (1945), 180–189; II, Ann. of Math. (2) **48** (1947), 472–494; III, Acta Math. **97** (1957), 267–308; MR **8**, 68; **9**, 35; **19**, 548], Paper I may be thought of as a survey of the results to be established in the series and Paper III as an introduction to the present paper.

In the equation of the title, k is assumed to be large and positive. $p(\varphi)$ is to be periodic with period 2π, have mean value 0 and $p(\pi+\varphi)=-p(\varphi)$. $p''(\varphi)$ is to be continuous. Let $p_1(\varphi)=\int p(\varphi)d\varphi$ have mean value 0. p_1 is to attain its upper and lower bounds only once in a period; $f(y)$ is to be an even function and $f''(y)$ is to be continuous; $f(y)$ shall have a single pair of zeros normalized to ± 1; $f'(1)>0$; and $f(y)$ is to have a positive lower bound in $y\geqq 2$. If $F(y)=\int_0^y f(y)dy$, then normalize $f(y)$ to make $F(-1)=2/3$. Thus the critical value for b is 2/3, as in the van der Pol equation, b always satisfies $0<b<2$. $g(y)$ is assumed to be odd and to have a continuous second derivative. Finally, it is assumed that $g'(y)\geqq 1$.

The one theorem of the paper is preceded by 33 lemmas. In this theorem two subsets B_1 and B_2 of real numbers are determined so that $B_1\cup B_2$ is properly contained in the interval $0<b<2$ and so that, if $b\in B_1$, then the equation has a family of trajectories each of which converges to one periodic trajectory, but if $b\in B_2$, then the equation has two families of trajectories such that each member of one family converges to one periodic trajectory while each member of the other family converges to a different periodic trajectory.

For the most part, the lemmas are meaningful only as steps in the proof of the theorem. However, the proof of Lemma 5, which is stated as Lemma 2 of Paper I, can be read independently and provides an interesting analysis of the Riccati equation, $dz/dx=z^2-x^2+1+\alpha-2\beta x$, when $z(0)=0$, $x\geqq 0$, $\alpha\geqq -1$, and, if $\alpha=-1$, $\beta<0$.

The results of the paper, based on joint work with M. L. Cartwright, are secured through subtle geometric and analytic arguments that are not easy to follow because of the overwhelming notational difficulties which, apparently, can not be avoided. *W. R. Utz* (Columbia, Mo.).

Krasovsky, N. N. On periodical solutions of differential equations involving a time lag. Dokl. Akad. Nauk SSSR (N.S.) **114** (1957), 252–255. (Russian)

Consider the systems

$$(1)\qquad \dot{y}_i=\sum q_{ij}(t)y_j(t-g_{ij}(t)),$$

$$(2)\qquad \dot{x}_i=\sum p_{ij}(t)x_j(t-h_{ij}(t))+\varphi_i(x_1(t-h_{i1}(t)),\cdots,t)+f_i(t),\quad (1\leqq i\leqq n)$$

where p_{ij}, q_{ij}, $\varphi_i(x,t)$, f_i are continuous and $g_{ij}(t)$, $h_{ij}(t)$ are piecewise continuous, periodic, non-negative and bounded. Suppose that $y=0$ of (1) is asymptotically stable [Èls'gol'c, Uspehi Mat. Nauk (N.S.) **9**, no. 4(62) (1954), 95–112; MR **17**, 44]. Th. 1: There are positive constants δ, γ, L such that, whenever $|p_{ij}-q_{ij}|<\delta$, $|h_{ij}-g_{ij}|<\gamma$, and φ_i is lipschitzian in x for L, (2) has a unique periodic solution which is asymptotically stable in Lyapunov's sense. Th. 2: For any $\varepsilon>0$ there is a $\delta>0$ such that, for $|\mu|<\delta$, the system

$$\dot{x}_i=\sum[q_{ij}(t)+\mu p_{ij}(t)]x_j(t-g_{ij}(t)-\mu h_{ij}(t))+f_i(t)+\mu\varphi_i(x_1(t-g_{i1}(t)-\mu h_{i1}(t)),\cdots,t)\quad (1\leqq i\leqq n$$

with φ_i lipschitzian in x and $g_{ij}+\mu h_{ij}\geqq 0$ has a periodic solution $x(t,\mu)$ satisfying $|x_i(t,0)-x_i(t,\mu)|<\varepsilon$.

H. A. Antosiewicz (Providence, R.I.).

Višik, M. I.; and Lyusternik, L. A. Stabilization of the solutions of certain differential equations in Hilbert space. Dokl. Akad. Nauk SSSR (N.S.) **111** (1956), 12–15. (Russian)

A family of trajectories $\{u=u(t)\}$ in a linear metric space M is said to stabilize around a path $v=v(t)$ if $\lim_{t\to\infty}\rho(u(t),v(t))=0$ for arbitrary $u\in\{u(t)\}$. The authors consider a differential equation $du/dt=A(t)\cdot u=f(t)$, $u|_{t=t_0}=u_0$, in a Hilbert space H, together with the related stationary equation $A(t)\cdot v=f(t)$. It is assumed that $A(t)$ is a linear operator with dense domain independent of t and such that $(A(t)u,u)\geqq\gamma(t)\cdot(u,u)$, $\gamma(t)>0$. This problem has wide interest especially when $A(t)$ is an elliptic differential operator with small parameter, i.e., $\lim A(t)$ is an operator of lower order.

Under the additional assumption that $A_t'(t)$ exists in the strong sense and is dominated by $A(t)$, $\|A_t'(t)v\|\leqq\delta(t)\|A(t)v\|$, sufficient conditions are derived in order that a solution u of the differential equation stabilize along a solution v of the stationary equation. For example, one sufficient condition is given by $\gamma(t)\geqq c^2>0$ and $\varepsilon(t)=\gamma(t)^{-1}\|f'(t)\|+\delta(t)\gamma(t)^{-1}\|f(t)\|=O(t^{-r})$, $r>0$. (There is a misprint in each of the remaining two sufficient conditions which can be easily corrected.) In the final section an estimate for $\|du/dt\|$ in terms of $\varepsilon(t)$ and $\gamma(t)-\delta(t)$ is obtained.

A. N. Milgram (Minneapolis, Minn.).

See also: Special Functions: Erdélyi and Swanson. **Banach Spaces, Banach Algebras, Hilbert Spaces:** Kostučenko. **Numerical Methods:** Valat; Fujita. **Mechanics of Particles and Systems:** Ziemba. **Fluid Mechanics, Acoustics:** Gil and Myshkis; Parsons.

Partial Differential Equations

Vishik, M. I.; and Lusternik, L. A. On elliptical equations containing small parameters in the terms with higher derivatives. Dokl. Akad. Nauk SSSR (N.S.) **113** (1957), 734–737. (Russian)

The authors construct an asymptotic solution $u_\varepsilon(x)$ for small ε to the boundary problem consisting of the equation $L_{r,\varepsilon}u=\sum_{s=p}^{r}\varepsilon^{s-p}L_s u=h$ and conditions

(1) $$u|_q=\cdots=(\partial^{k-1}u/\partial n^{k-1})|_q=0,$$

(2) $$(\partial^k u/\partial n^k)|_q=\cdots=(\partial^{k+l-1}u/\partial n^{k+l-1})|_q=0,$$

where $L_s u$ and $L_{r,\varepsilon}u$ are elliptic operators with $p=2k$, $r=2(k+l)$, and q is the boundary of the region in the space $(x_1,\cdots,x_n)$ in which the solution is sought. Considering the difference $v_\varepsilon(x)=u_\varepsilon(x)-w(x)$, where $w(x)$ is a solution of the equation $L_p w=h$, to be a boundary layer, they introduce local coordinates in the neighborhood of q which transform the original problem for u_ε into a sequence of boundary problems for the functions $v_0, v_1, \cdots, v_N$ such that $v_\varepsilon=v_0+\varepsilon v_1+\cdots+\varepsilon^N v_N$. For $N=k$, v_ε satisfies the equation $L_{r,\varepsilon}v=O(\varepsilon)$, while $v_\varepsilon+w$ satisfies (1) exactly and (2) to within ε. To v_ε then is added $\varepsilon\alpha$, where α depends upon the local coordinates, so that $v_\varepsilon+\varepsilon\alpha$ satisfies (2). The solution of the original problem finally has the form

$$u_\varepsilon=(w_0+v_\varepsilon+\varepsilon\alpha_0)+\cdots+\varepsilon^s(w_s+v_{s\varepsilon}+\varepsilon\alpha_s)+\beta_s,$$

where $L_{r,\varepsilon}(\beta_s)=O(\varepsilon^{s+1})$. The actual result obtained solves a slightly more general problem.

R. N. Goss.

Amanov, T. I. On the solution of the biharmonic problem. Dokl. Akad. Nauk SSSR (N.S.) **113** (1957), 727–730. (Russian)

The author studies the relation between interior and boundary norms associated with solutions of $\Delta^2 u=0$ defined in the unit disc σ. He proves that if $\lim_{\rho\to 1}u(\rho,\theta)=\varphi(\theta)$, $\lim_{\rho\to 1}\partial u/\partial\rho=\psi(\theta)$, and if all derivatives of u of order up to m are square summable on σ, then $\varphi\in H_2^{(m-1)}(M_1)$ and $\psi\in H_2^{(m-3/2)}(M_2)$ for suitable constants M_1, M_2. The function classes $H_2^{(r)}(M)$ are those introduced by Nikolskiĭ [Trudy Mat. Inst. Steklov. **38** (1951), 244–278; MR **14**, 32]. The author also proves the stability of a solution in the sense that if $\varphi(\theta)$ and $\psi(\theta)\to 0$ in norms associated with $H_2^{(m-1+\varepsilon_1)}(M_1)$ and $H_2^{(m-3/2+\varepsilon_2)}(M_2)$, respectively, then the integral over σ of the sum of squares of mth order derivatives of u also tends to zero. The proofs use methods of Fourier analysis.

R. Finn.

Bicadze, A. V. On the uniqueness of solution of the problem of Frankl' for Čaplygin's equation. Dokl. Akad. Nauk SSSR (N.S.) **112** (1957), 375–376. (Russian)

In an earlier article [same Dokl. (N.S.) **109** (1956), 1091–1094; MR **18**, 743] the author considered the problem of Frankl' [Prikl. Mat. Meh. **20** (1956), 196–202; MR **18**, 255] for the equation of Lavrentieff:

$$U_{xx}+(\operatorname{sgn} y)U_{yy}=0.$$

In the present article he considers the same problem for the equation of Čaplygin

$$k(y)u_{xx}+u_{yy}=0,$$

where $k(0)=0$, $k'(y)>0$, $k(-y)=-k(y)$.

From the introduction.

Henrici, Peter. On the domain of regularity of generalized axially symmetric potentials. Proc. Amer. Math. Soc. **8** (1957), 29–31.

Analytic solutions of $u_{xx}+u_{yy}+\frac{p}{y}u_y=0$ were shown by the reviewer, extending results of A. Weinstein, to be uniquely determined by $u(x,0)$ (necessarily analytic), provided $p\neq 0, -1, -2, \cdots$ [Weinstein, Trans. Amer. Math. Soc. **63** (1948), 342–354; MR **9**, 584; Hyman, Nederl. Akad. Wetensch. Proc. Ser. A. **57** (1954), 408–413; MR **16**, 368]. The author considers the following question which he answers with a broad theorem: for x, y complex, how is the domain of analyticity of the solution $u(x,y)$ related to the domain of analyticity of $u(x,0)$? Earlier results along this line were obtained by the reviewer [see reference above] and Erdélyi [Comm. Pure Appl. Math. **9** (1956), 403–414; MR **18**, 800].

M. A. Hyman (Yorktown Heights, N.Y.).

Bers, Lipman. Formal powers and power series. Comm. Pure Appl. Math. **9** (1956), 693–711.

This paper is a further contribution to the theory of pseudo-analytic functions which can be regarded as the function theory associated with a linear, homogeneous, elliptic differential equation of the form

(*) $$a_{11}(x,y)\varphi_{xx}+2a_{12}(x,y)\varphi_{xy}+a_{22}(x,y)\varphi_{yy}+a_1(x,y)\varphi_x+a_2(x,y)\varphi_y+a_0\varphi=0.$$

This function theory is entirely similar in spirit to that associated with Laplace's equation and even, in part, can be reduced to the latter in the cases in which the a_{ij} and the a_i are sufficiently smooth [cf. Bers, Theory of pseudo-analytic functions, Inst. Math. Mech., New York Univ., 1953; Contributions to the theory of partial differential equations, Princeton, 1954, pp. 69–94; MR **15**, 211; **16**, 1114]. Because of its importance for nonlinear equations,

the author, however, sacrificing the direct connection to classical function theory, has essayed to build the theory of pseudo-analytic functions anew under essentially minimal smoothness conditions, which for (*) would be that the a_{ij} satisfy a Hölder condition and the a_i belong to L_p, $p>2$, within some disk. As usual in this theory, a second order equation such as (*) is not considered directly but is reduced to a first order equation with Hölder-continuous complex coefficients for a complex-valued dependent variable $w(x, y)\equiv w(z)$, $z=x+iy$, called "pseudo-analytic". The attempt to minimize the smoothness conditions was initiated in a previous paper [Lectures on functions of a complex variable, Univ. of Michigan Press, 1955, pp. 213–244; MR 17, 837] dealing predominantly with local problems, among them the character of the mapping from the z-plane into the w-plane and the asymptotic description of single-valued pseudo-analytic functions at zeros and at isolated singularities. The present paper is devoted to global problems under similarly relaxed conditions on the coefficients which now, however, are presumed to be defined in the whole plane and to behave suitably at ∞. An infinite set of equations, distinguished by the index ν varying over all the integers, are introduced and for each of these a set of particular solutions $Z_\nu^{(n)}(a, z, \zeta)$, $n=0, \pm1, \pm2, \cdots$, which, in many respects, are analogous to $a(z-\zeta)^n$. These particular solutions figure in integral formulas of Cauchy's types and also in expansions, analogous to those of Taylor, Laurent, and Runge, for pseudo-analytic functions, both single- and multi-valued. It is emphasized that these particular solutions do not depend upon the domain in which a given pseudo-analytic function to be expanded is defined but only on the point ζ in the neighborhood of which the expansion is to take place. *A. Douglis.*

Browder, Felix E. On the regularity properties of solutions of elliptic differential equations. Comm. Pure Appl. Math. **9** (1956), 351–361.

The author states a number of results concerning variational boundary value problems for an elliptic equation (*) $Au=f$ of order $2m$ in a bounded domain G in n-space. Proofs are to appear elsewhere, though some are sketched. By means of Hilbert space theory it can be shown that, under suitable assumptions, the Fredholm alternative and the usual results concerning eigenvalues and eigenfunctions are valid for such a "variational", or "generalized" boundary value problem. If a ("generalized") solution u to such a problem is sufficiently differentiable in $\bar{G}$ it becomes a classical solution to (*) in G and satisfies (pointwise) a number of linear conditions on the boundary derivatives. The main result of the paper is that if f and the coefficients of A are sufficiently regular this situation is achieved. Detailed statements concerning regularity of u in the interior and at the boundary are given. In deriving the regularity of u at the boundary the author combines a technique used by Nirenberg (of estimating the L^2 norms of tangential derivatives of u by taking finite difference quotients) with Aronszajn's lemma on coercive forms to estimate the L^2 norm of the $2m$th derivatives of u in terms of the L^2 norms of f and u.

W. Littman (Berkeley, Calif.).

Browder, Felix E. The asymptotic distribution of eigenfunctions and eigenvalues for semi-elliptic differential operators. Proc. Nat. Acad. Sci. U.S.A. **43** (1957), 270–273.

L'A. dà un'interessante e larga generalizzazione del teorema sulla distribuzione degli autovalori e delle autofunzioni per operatori differenziali, mediante un metodo già adoperato dallo stesso A. [C. R. Acad. Sci. Paris **236** (1956), 2140–2142; MR **15**, 320] e analogo ad uno di L. Gårding [Kungl. Fysiogr. Sällsk. i Lund Förh. **21** (1951), no. 11; MR **14**, 653]. Sia $D_j=i^{-1}\partial/\partial x_j$, $D^\alpha=D_{\alpha_1}\cdots D_{\alpha_k}$ con $\alpha=(\alpha_1, \cdots, \alpha_k)$, $|\alpha|=k$, $\xi=(\xi_1, \cdots, \xi_n)$, $\xi^\alpha=\xi_{\alpha_1}\cdots\xi_{\alpha_k}$, $m=(m_1, \cdots, m_n)$, m_j interi positivi, $\xi_m{}^\alpha=(\xi_{\alpha_1})^{m_{\alpha_1}}\cdots(\xi_{\alpha_k})^{m_{\alpha_k}}$, $|\alpha|_m=\sum_{j=1}^k m_{\alpha_j}$, G un aperto dello spazio euclideo E^n, $C_c^\infty(G)$ la classe delle funzioni indefinitamente differenziabili con supporto compatto in G. Sia poi $A=\sum a_\alpha(x)D^\alpha$ un operatore differenziale, con $a_\alpha(x)\in C^\infty(G)$, formalmente autoaggiunto (cioé $(A\varphi, \psi)=(\varphi, A\psi)$ per $\varphi, \psi\in C_c^\infty(G)$, $(\ ,\ )$ prodotto scalare in $L^2(G)$). Si dice che A é semiellittico nel punto x_0 rispetto al vettore $(m_1, \cdots, m_n, q)$ di interi positivi, se esiste un intorno N di x_0 e un C^∞-sistema di coordinate in $\bar{N}$, $\{y_j\}$, tale che 1) $A=\sum_{|\alpha|_m\le q} a_\alpha'(y)D_y{}^\alpha$ per y in N, 2) posto $A_0(y, \xi)=\sum_{|\alpha|_m\le q} a_\alpha'(y)\xi_m{}^\alpha$, allora $A_0(y, \xi)$ é una forma di grado q in ξ definita positiva. Supposto allora A formalmente autoaggiunto e semiellittico e consideratolo come un operatore in $L^2(G)$ con domino $C_c^\infty(G)$, esso é simmetrico e limitato inferiormente; esso ammette allora un prolungamento autoaggiunto A_1 in $L^2(G)$; sia $e_{x,\lambda}$ la funzione spettrale di A_1. Si ha il teorema: Per ogni x, $y\in G$, al tendere di λ a $+\infty$, si ha $e_{x,\lambda}(y)=(2\pi)^{-n}w_{A_\lambda}(x)\{\delta_{x,y}+o(1)\}\lambda^b$, dove $b=q^{-1}(\sum_{j=1}^n m_j)$, $w_{A_\lambda}(x)=\int_{A_\lambda(x,\xi)<1} d\xi$, $\delta_{x,y}=1$ se $x=y$, $\delta_{x,y}=0$ se $x\neq y$.

Particolarmente notevole é l'applicazione a tutti quei problemi al contorno ellittici di cui recentemente l'A. ha stabilito la regolarità anche alla frontiera delle soluzioni [v. il lavoro recensito sopra; v. anche L. Nirenberg, Comm. Pure Appl. Math. **8** (1955), 649–675; MR **17**, 742]. Si ha allora, detti λ_i gli autovalori,

$$\sum_{\lambda_i\le t} 1=\{1+o(1)\}(2\pi)^{-n}\Big\{\int w_{A_\lambda}(x)dx\Big\}t^b,$$

per $t\to+\infty$. *E. Magenes* (Genova).

Sexl, Th. Zur systematischen Integration der Laplaceschen Differentialgleichung. Österreich. Ing.-Arch. **10** (1956), 280–288.

Boyarskiĭ, B. On a boundary problem for a system of elliptic first-order partial differential equations. Dokl. Akad. Nauk SSSR (N.S.) **102** (1955), 201–204. (Russian)

Let T be a region of the complex z plane bounded by the curve L. Continuous solutions $U(z)$ are sought for the equation

$$\partial U/\partial\bar{z}=A\bar{U},$$

where $A(z)$ is a given complex-valued function.

The article falls into two parts. In the second part $U(z)$ must satisfy the boundary condition

$$\mathrm{Re}\,[aU+\int_L K(t, t_0)U(t)ds]=f(t_0),$$

where $a(s)$ is a given function of the arc-length s on L, and $t=t(s)\in L$. In the first part the boundary conditions are somewhat more complicated.

Koševlev, A. I. On differentiability of solutions of elliptic differential equations. Dokl. Akad. Nauk SSSR (N.S.) **112** (1957), 806–809. (Russian)

Let Ω be an open, bounded set in $x=(x_1, \cdots, x_n)$ space, and Γ be its boundary. Suppose that the functions

$a_{ik}(x)$ $(i, k=1, \cdots, n)$ are twice continuously differentiable on $\Omega+\Gamma$ and satisfy the ellipticity condition

$$\sum_{i,k=1}^{n} a_{ik}(x)\xi_i\xi_k \geqq \mu \sum_{i=1}^{n} \xi_i^2 \quad (x \in \Omega),$$

where μ is a positive constant. Consider the boundary value problem consisting of the elliptic equation

$$L(u) \equiv \sum_{i,k=1}^{n} \frac{\partial}{\partial x_i}\left(a_{ik}(x)\frac{\partial u}{\partial x_k}\right) = f(x), \text{ in } \Omega,$$

plus the boundary condition $u|_\Gamma=0$. J. Schauder [Math. Z. 38 (1934), 257–282] showed that if f satisfies a Lipschitz condition with exponent α with $0<\alpha<1$ on $\Omega+\Gamma$, then the second derivatives of the solution u of the boundary value problem also have the same property. O. A. Ladyženskaya [Dokl. Akad. Nauk SSSR (N.S.) 79 (1951), 723–725; MR 14, 280] showed that if $f \in L_2(\Omega)$ then the second derivatives (in the generalized sense of S. L. Sobolev) are also in $L_2(\Omega)$. The present paper is concerned with the case when $f \in L_p(\Omega)$ where $p>n/2$. Theorem: If $f \in L_p(\Omega)$, $p>n/2$, then there exists a generalized solution of the boundary value problem which satisfies the inequality

$$\|u\|_{W_p^{(2)}(\Omega)} \leqq C\|f\|_{L_p(\Omega)},$$

where the constant C is independent of u and of f. ($W_p^{(2)}$ denotes the space of functions possessing generalized derivatives which are in $L_p(\Omega)$, following S. L. Sobolev [Some applications of functional analysis in mathematical physics, Izdat. Leningrad. Gos. Univ., 1950; MR 14, 565].)

J. B. Diaz (College Park, Md.).

Artemov, G. A. Application of Čaplygin's method to the solution of the characteristic Cauchy problem for a partial differential equation of parabolic type. Dokl. Akad. Nauk SSSR (N.S.) 112 (1957), 791–792. (Russian)

In an earlier article [same Dokl. (N.S.) 102 (1955), 197–200; MR 17, 91] the author established a theorem analogous to Čaplygin's [New methods of approximate integration of differential equations, Gostehizdat, Moscow-Leningrad, 1950], for the equation

$$u_{xy}=f(x, y, u, p, q)$$

with the initial condition $u|_l=u(t)$, $p|_l=p(t)$, $q|_l=q(t)$ along a non-characteristic curve l. In the present article he extends the theorem to the case where $u(x, y)$ is given along the characteristics $x=x_0$, $y=y_0$.

From the introduction.

Il'in, V. A. On the foundations of the Fourier method for the wave equation. Uspehi Mat. Nauk (N.S.) 12 (1957), no. 4(76), 289–296. (Russian)

In N-dimensional $x=(x_1, \cdots, x_N)$ space, consider a bounded open set G, whose boundary is a surface Γ, and the following mixed boundary value problem for the wave equation: $\Delta v-a^{-2}v_{tt}=-f(x, t)$, in G; $v(x, 0)=\varphi(x)$, $v_t(x, 0)=\psi(x)$, for x in G; $v|_\Gamma=0$. The formal solution of this problem by Fourier's method leads to the series

$$(*)\quad v(x, t)=\sum_{n=1}^{\infty} u_n(x)\cdot\left\{\varphi_n \cos a\lambda_n^{\frac{1}{2}}t+\frac{\psi_n}{a\lambda_n^{\frac{1}{2}}}\sin a\lambda_n^{\frac{1}{2}}t\right\} + \sum_{n=1}^{\infty} u_n(x)a\int_0^t f_n(\tau)\sin a\lambda_n^{\frac{1}{2}}(t-\tau)\,\frac{d\tau}{\lambda_n^{\frac{1}{2}}},$$

where $u_n(x)$ are the orthonormalized eigenfunctions of the eigenvalue problem $\Delta u+\lambda u=0$, in G; $u|_\Gamma=0$; and φ_n, ψ_n, and $f_n(t)$ are the Fourier coefficients of the functions $\varphi(x)$, $\psi(x)$, and $f(x, t)$ with respect to the u_n. Under suitable differentiability hypotheses on φ, ψ, and f, the author shows that the series (*) is a solution of the boundary value problem in the classical sense (i.e., the series and the series obtained by differentiating it twice termwise are uniformly convergent on any closed subset interior to the cylinder consisting of all points (x, t), where x is in $G+\Gamma$, and $0\leqq t\leqq l$); the surface Γ is only assumed to be a Liapunov surface. This extends the results of O. A. Ladženskaya [The mixed problem for a hyperbolic equation, Gostehizdat, Moscow, 1953; MR 17, 160], who made the same assumptions about the functions φ, ψ, and f, but used more stringent differentiability restrictions on the surface Γ. The author also treats the boundary value problem involving a boundary condition of the second or third kind: $[\partial v/\partial \nu+h(s)v]_\Gamma=0$, where $h(s)\geqq 0$.

J. B. Diaz (College Park, Md.).

Blondel, Jean-Marie. Perturbation singulière pour une équation du second ordre, linéaire et hyperbolique. C. R. Acad. Sci. Paris 245 (1957), 1496–1498.

On étudie, pour $\varepsilon \to +0$, le comportement de la solution d'un problème de Cauchy, et aussi celui de la solution qui se réduit à deux fonctions données, sur deux caractéristiques, pour l'équation: $\varepsilon H\{z\}+L\{z\}+cz=0$; où H=opérateur différentiel linéaire hyperbolique du second ordre; L=opérateur différentiel linéaire du premier ordre; $c(x, y)$=fonction donnée. (Résumé de l'auteur.)

R. McKelvey (Boulder, Colo.).

Phillips, R. S. Dissipative hyperbolic systems. Trans. Amer. Math. Soc. 86 (1957), 109–173.

Consider the system of equations $Ey_t=(Ay)_x+By$ defined in the region $-\infty\leqq a<x<b\leqq\infty$, $0<t$, where y is a k-dimensional complex vector-valued function of x and t; E, A and B are $k\times k$ matrix-valued functions of x; E is hermitian and positive definite, and A is hermitian and of constant rank r; and the elements of E and A are absolutely continuous on every compact subinterval of (a, b), while those of E_x, A_x and B are square integrable on compact subintervals. This is a hyperbolic system, and if we indicate the k-dimensional vector space inner product by (y, z), the energy integral of our system is one half the integral $\int_a^b (Ey, y)dx$. It is thus natural to introduce the hilbert space H of k-dimensional vector-valued functions defined on the interval (a, b) with the inner product $\langle y, z\rangle=\int_a^b (Ey, z)dx$. Any solution of our system then formally satisfies

$$\langle y, y\rangle_t=[(Ay, y)^b-(Ay, y)^a]+\int_a^b ((B+B^*+A_x)y, y)dx.$$

A dissipative Cauchy problem is essentially one for which this derivative is not positive. More precisely, the system itself is said to be dissipative if $B+B^*+A_x\leqq\Theta$, and the imposed boundary conditions are dissipative if they imply $(Ay, y)^b-(Ay, y)^a\leqq 0$.

This paper studies the Cauchy problem for such dissipative systems and characterizes all possible dissipative boundary conditions which lead to well posed problems in the sense of semi-group theory. Let $Ly=E^{-1}[(Ay)_x+By]$, and let L_1 be the operator on H which is formally L and whose domain consists of all those y for which (Ay, y) exists at both end points. Precisely stated, the problem is then to find all possible dissipative restrictions of L_1 which are the infinitesimal generators of strongly continuous semi-groups of bounded linear operators on H. The attack on this problem is carried out through the con-

struction of an explicit representation for the resolvent of L. The final characterization is too complicated to present here, but it involves subspaces of certain finite-dimensional quotient spaces of the domain of L_1. These quotient spaces are defined in terms of the elements belonging to the domain of the operator dual to L_1, and simultaneously with L_1 this dual operator and its restrictions must be examined in detail.

The boundary conditions which are obtained by this process do not include those of elastic type. In order to obtain these also, the paper concludes by extending the original system to one involving a coupling of the two end points to simple mechanical systems. Again all possible dissipative boundary conditions which lead to well posed Cauchy problems are characterized under the further assumption that energy is also dissipated in the mechanical systems. One of the very interesting features here is that at the end points the "values" of a solution are considered to be points in the quotient spaces mentioned above, and therefore the mathematical description of the mechanical coupling involves linear maps from these quotient spaces into the spaces describing the mechanical systems. *G. Hufford* (Stanford, Calif.).

Zhdanovich, W. F. Solution by the Fourier method of non-self-adjoint mixed problems for hyperbolic systems on a plane. Dokl. Akad. Nauk SSSR (N.S.) **114** (1957), 934–937. (Russian)

The hyperbolic system

$$\frac{\partial}{\partial t}u(x,t)=A(x)\frac{\partial}{\partial x}u(x,t)+B(x)u(x,t) \qquad (0\leqq x\leqq l,\ 0\leqq t\leqq T<\infty),$$

with boundary conditions

$$M\frac{\partial}{\partial t}u(0,t)+Nu(0,t)+P\frac{\partial}{\partial t}u(l,t)+Qu(l,t)=0,$$

$$u(x,0)=f(x),$$

is solved by assuming for the vector $u(x,t)$ the form $y(x)e^{\lambda t}$. A series of theorems establishes the precise conditions for validity of the solution. *R. N. Goss.*

Slobodeckiĭ, L. N.; and Hramova, M. I. On the uniqueness of the solution of the Cauchy problem for quasi-linear symmetric systems of differential equations. Uspehi Mat. Nauk (N.S.) **11** (1956), no. 4(70), 155–162. (Russian)

The authors prove the uniqueness of the solution of the Cauchy problem for quasi-linear "symmetric" systems of partial differential equations of the first and second order under the following conditions.

In the case of the 1st order "symmetric" system

$$\frac{\partial u}{\partial t}=\sum_{i=1}^{n}A_i\frac{\partial u}{\partial x_i}+f \text{ defined on a region } D$$

($A_i=\|a_{jk}^{(i)}\|$ and $u=(u_1, u_2, \cdots, u_N)$, $f=(f_1, f_2, \cdots, f_N)$, where $a_{jk}=a_{jk}(t,x,u)$ and $f_k=f_k(t,x,u)$ are complex functions of the real variables t and $x=(x_1, x_2, \cdots, x_n)$ and the unknown vector function u; the symmetry of the system implies that the matrices A_i are hermitian), the uniqueness can be shown if 1) $a_{jk}^{(i)}(t,x,u)$ are continuous functions with continuous partial derivatives in $x_1, \cdots, x_n, u_1, \cdots, u_N$ on D, and 2) the vector function $f=(f_1, f_2, \cdots, f_N)$ satisfies the inequality:

$|f(t,x,u)-f(t,x,v)|\leqq\phi(|u-v|)$ for (t,x,u) and $(t,x,v)\in D$,

where $F(z)=z^1\phi(z^1)$ is a positive, non-decreasing and non-convex function such that $\int_0^\delta dz/F(z)$ diverges ($\delta>0$).

For the second-order "symmetric" system

$$\frac{\partial^2 u}{\partial t^2}=\sum_{i,j=1}^{n}A_{ij}\frac{\partial^2 u}{\partial x_i\partial x_j}+f\left(t,x,u,\frac{\partial u}{\partial t},\frac{\partial u}{\partial x_1},\cdots,\frac{\partial u}{\partial x_n}\right)$$

($A_{ij}=A_{ij}(t,x,u,\partial u/\partial t,\partial u/\partial x_1,\cdots,\partial u/\partial x_n)$ are hermitian matrices, $A_{ij}=A_{ji}$, such that, for any vectors $\xi_1, \cdots, \xi_n$,

$$\sum_{i,j=1}^{n}\xi_iA_{ij}\xi_j\geqq\mu^2\sum_{i=1}^{n}|\xi_i|^2 \quad (\mu=0);$$

the other notation is the same as for the first case), the uniqueness can be shown if 1) the elements of the matrices A_{ij} are continuous and continuously differentiable in the variables $x_1, \cdots, x_n, u_1, \cdots, u_N, \partial u_1/\partial t, \cdots, \partial u_N/\partial t, \partial u_1/\partial x_1, \cdots, \partial u_N/\partial x_n$, and 2) the vector function $f(t,x,u,u^{(0)},u^{(1)},\cdots,u^{(n)})$ satisfies the inequality

$$|(t,fx,u,u^{(0)},\cdots,u^{(n)})-f(t,x,v,v^{(0)},\cdots,v^{(n)})| \leqq\phi\left(\left[|u-v|^2+\sum_{k=0}^{n}|u^{(k)}-v^{(k)}|^2\right]^{1/2}\right),$$

where $F(z)=z^1\phi(z^1)$ has the same properties as in the first case. *U. W. Hochstrasser* (Lawrence, Kansas).

Itô, Seizô. A boundary value problem of partial differential equations of parabolic type. Duke Math. J. **24** (1957), 299–312.

This paper extends the results of two previous papers by the same author [Osaka Math. J. **5** (1953), 75–92; **6** (1954), 167–185; MR **15**, 36; **16**, 370] to obtain the unique solution of the non-homogeneous parabolic equation

$$a^{ij}(t,x)\frac{\partial^2 u}{\partial x_i\partial x_j}+b^i(t,x)\frac{\partial u}{\partial x_i}+c(t,x)u-\frac{\partial u}{\partial t}=h(t,x)$$

subject to non-homogeneous boundary conditions on a domain $[s_0,t_0]\times\bar{D}$, where $\bar{D}$ is the closure of a domain in m-dimensional space bounded by a finite number of $(m-1)$-dimensional hypersurfaces of class c^3. The matrix $\|a^{ij}(t,x)\|$ is strictly positive definite symmetric, the functions a^{ij}, b^i, c are subject to certain differentiability conditions and $h(t,x)$ is continuous. An explicit form of the solution is given in terms of the fundamental solution of the parabolic equation which was obtained by the methods of the previous work cited above.

C. G. Maple (Ames, Iowa).

Smirnov, M. M. On a boundary problem for an equation of mixed type. Vestnik Leningrad. Univ. **12** (1957), no. 1, 80–96, 209–210. (Russian. English summary)

The partial differential equation of mixed type considered is

$$(*)\qquad \frac{\partial^4 u}{\partial x^4}+2\operatorname{sgn} y\frac{\partial^2 u}{\partial x^2\partial y^2}+\frac{\partial^4 u}{\partial y^4}=0.$$

The domain under consideration is the plane domain D bounded above by a simple smooth arc σ, lying except for its end points $A(0,0)$ and $B(1,0)$ in the upper half plane $y>0$; and bounded below by the characteristics of (*): AC: $y=-x$ and BC: $y=x-1$. The boundary value problem in question consists in the determination of a solution of (*) in D which fulfills the following boundary conditions:

$$u|_\sigma=\varphi_1(s),\ \left.\frac{\partial u}{\partial n}\right|_\sigma=\varphi_2(s),\ \left.\frac{\partial u}{\partial n}\right|_{y=-x}=\psi_3(x)\ (0\leqq x\leqq\tfrac{1}{2}),$$

$$\left.\frac{\partial u}{\partial n}\right|_{y=x-1}=\psi_4(x)\ (\tfrac{1}{2}\leqq x\leqq 1),$$

where n is the inner normal. Under suitable differentiability conditions, existence and uniqueness are proved. Existence is obtained by employing the singular integral equation theory of D. I. Šerman [C. R. (Dokl.) Acad. Sci. URSS (N.S.) **28** (1940), 28–31; MR **2**, 270], N. I. Muskelišvili [Singular integral equations, OGIZ, Moscow-Leningrad, 1946; MR **8**, 586; **15**, 434] and N. P. Vekua [Systems of singular integral equations and some boundary problems, Gostehizdat, Moscow-Leningrad, 1950; MR **13**, 247]. *J. B. Diaz* (College Park, Md.).

Barantsev, R. G. A boundary problem for equation $\psi_{\sigma\sigma}-K(\sigma)\psi_{\theta\theta}=0$ with values given on the characteristic and σ=const. lines. Dokl. Akad. Nauk SSSR (N.S.) **113** (1957), 955–958. (Russian)

The problem stated more fully is that of solving the equation in the title for $K(\sigma)>0$ in the strip $S[\sigma_0\leqq\sigma\leqq\sigma_1, -\infty<\theta<\infty]$ with conditions ψ=constant on the lines $\sigma=\sigma_0$, $\sigma=\sigma_1$, and $\psi=\bar{\psi}(\sigma)$ on a characteristic segment between the lines. Under a change of variable the equation becomes $v_{\xi\xi}-v_{\theta\theta}+N(\zeta)v=0$, with data $v=p(\zeta)$ on $\theta=\zeta$, and $v=0$ on $\zeta=0$ and $\zeta=1$. Application of the Fourier method results in a solution having the form

$$(*)\qquad v=\sum_{n=1}^{\infty} B_n(\zeta)(a_n\cos\mu_n\theta-b_n\sin\mu_n\theta)/\mu_n,$$

where μ_n and B_n are the eigenvalue and eigenfunctions, respectively, of the Sturm-Liouville problem $B_n''+[\mu_n^2+N(\zeta)]B_n=0$, $B_n(0)=B_n(1)=0$, and a_n, b_n denote suitable integrals. Initially it is assumed that $N(\zeta)$ and $p(\zeta)$ are continuous on [0, 1] and that $p(\zeta)$ is of bounded variation there, approaching zero at the ends of the interval; under these conditions (*) is a "generalized" solution. The additional conditions necessary to make (*) a solution in the classical sense are obtained. *R. N. Goss.*

Barantsev, R. G. A mixed problem for equation $\psi_{\sigma\sigma}-K(\sigma)\psi_{\theta\theta}=0$ with Cauchy data given on curve $\theta=s(\sigma)$. Dokl. Akad. Nauk SSSR (N.S.) **114** (1957), 919–922. (Russian)

The boundary problem of the paper reviewed above is altered as indicated in the present title, where $\theta=s(\sigma)$ is not a characteristic. By the same change of variable and the same method one obtains

$$(**)\qquad v=\sum_{n=-\infty}^{\infty}{}' c_nB_n(\zeta)\exp(-i\lambda_n\theta),$$

where now v satisfies the conditions $v=0$ on $\zeta=0$ and $\zeta=1$, $v=p(\zeta)$ on $\theta=l(\zeta)$, $v_\theta=q(\zeta)$ on $\theta=l(\zeta)$; $B_n(\zeta)$ are eigenfunctions of the same Sturm-Liouville problem with μ_n replaced by λ_n; and $\lambda_{-n}=-\lambda_n$, $B_{-n}(\zeta)=-B_n(\zeta)$. Conditions under which $p(\zeta)$ and $q(\zeta)$ can be expanded in terms of the eigenfunctions are deduced, and conditions under which (**) represents a generalized or the classical solution are discussed. *R. N. Goss* (San Diego, Calif.).

Šabat, B. V. Examples of solution of the Dirichlet problem for equations of mixed type. Dokl. Akad. Nauk SSSR (N.S.) **112** (1957), 386–389. (Russian)

The Dirichlet problem is considered for the following three equations, occurring in the theory of the flow of gas through a nozzle:

$$(1)\qquad \partial^2u/\partial x^2-\partial^2u/\partial y^2=0;$$

$$(2)\qquad \partial^2u/\partial x^2+\operatorname{sgn} y\,\partial^2u/\partial y^2=0;$$

and, in polar coordinates ρ and t,

$$(3)\qquad \rho^2\partial^2u/\partial\rho^2+\rho\partial u/\partial\rho+\operatorname{sgn}(1-\rho)\partial^2u/\partial t^2=0;$$

in each case with suitable boundary conditions.

An example of the theorems proved is: if in (3) the values of u are assigned on the circumference $\rho=h>1$, then there exists an everywhere dense set of values of $h>1$ for which the Dirichlet problem has a unique solution for arbitrary (twice differentiable) boundary values, and an everywhere dense set for which this problem either has no solution or continuum-many solutions.

Pliś, A. On characteristics of partial differential equations. Bull. Acad. Polon. Sci. Cl. III. **5** (1957), 957–958, LXXX. (Russian summary)

The author weakens assumptions for the existence of solutions of a system of characteristic equation derived from a first order partial differential equation.

M. Steinberg (Culver City, Calif.).

See also: Harmonic Functions, Convex Functions: Mařík. **Ordinary Differential Equations:** Egorov; Višik and Lyusternik. **Differential Geometry:** Nitsche. **Numerical Methods:** Bellman; Ustinova; Ventcel'; Kantorovič, Krylov and Černin; Vzorova.

Difference Equations, Functional Equations

★ **Рябенький, В. С.; и Филиппов, А. Ф. [Ryaben'kiĭ, V. S.; and Filippov, A. F.] Об устойчивости разностных уравнений. [On stability of difference equations.]** Gosudarstv. Izdat. Tehn.-Teor. Lit., Moscow, 1956. 171 pp. 4.25 rubles.

This booklet gives a clear and comprehensive study of the stability of difference equations approximating differential equations (ordinary and partial), making use of the methods of functional analysis. In the first chapter the convergence of the solution of the difference equation to the solution of the differential equation is investigated. The author introduces here the notion of correctness of the difference equations and the corresponding boundary conditions. [A system of difference equations $R_h u_h=f$ and boundary conditions $r_{hi}(u_h)=\varphi_{hi}$ $(i=1, 2, \cdots, s)$ on a net D_h with meshlength h is correct if for sufficiently small h a solution exists and depends continuously on the right sides f and φ_{hi}, the dependence being uniformly continuous in h.] It is shown that under certain conditions the convergence to the solution of the differential equation follows from the correctness of the system of difference equations.

In Chapter 2 the different types of stability and their connections with the stability in the initial conditions are considered. The last paragraphs of this chapter consider also the stability of iterative procedures for solving the difference equations. In Chapter 3 some of the well-known conditions for stability are given, e.g., the investigation of the stability by studying the growth of a unit error, the use of the method of separation of variables to derive the characteristic equation whose roots determine the stability, etc. The end of the chapter is devoted to the study of the influence of the method of approximation of the boundary conditions and of other factors on the stability. Chapter 4 is devoted to the investigation of the stability of difference equations with respect to the initial conditions in those cases where the method of separation of variables can be used. A criterion for the solvability of the difference equations as well as for their stability with respect to the initial conditions and the given right-hand sides of the differential equations is derived. Necessary and sufficient algebraic conditions for the stability are

given. Finally, difference equations are constructed which are stable and approximate some of the well-known differential equations.

In the last chapter the method of separation of variables is applied to the investigation of the stability with respect to the initial conditions of systems of difference equations approximating systems of differential equations of the parabolic and hyperbolic type. In particular, the Cauchy problem is considered, making use of the methods and results of a study by I. G. Petrovsky [Bull. Univ. d'Etat Moscou Ser. Internat. Sect. A. **1** (1938), no. 7].

U. W. Hochstrasser (Lawrence, Kansas).

Koval, P. I. Sur la stabilité des solutions des systèmes des équations linéaires aux différences finies. Ukrain. Mat. Ž. **9** (1957), 141–154. (Russian. French summary)

The author considers vector-matrix difference equations of the form $x_{n+1}=A_nx_n$ $(n=1, 2, \cdots)$, and shows that all solutions tend to zero as n increases provided that all characteristic roots of the limit matrix $A=\lim A_n$ are less than one in absolute value. The results are extensions of earlier results of Perron, Ta Li, Spath and the reviewer. *R. Bellman* (Santa Monica, Calif.).

Ghermănescu, M. Sur la définition fonctionnelle des fonctions trigonométriques. Publ. Math. Debrecen **5** (1957), 93–96.

The author develops a simple method for solving the functional equations

(1) $f(x)+f(y)=f\{xy-(1-x^2)^{\frac{1}{2}}(1-y^2)^{\frac{1}{2}}\}$,

(2) $f(x)+f(y)=f\{xy+(x^2-1)^{\frac{1}{2}}(y^2-1)^{\frac{1}{2}}\}$.

Continuous monotone solutions of these equations are, for (1), $f(x)=k$ arc cos x and, for (2), $f(x)=k$ arc cosh x [Aczél and Varga, same Publ. **4** (1955), 3–15; MR **17**, 777].

Starting with the equation

(3) $g(u+v)=g(u)g(v)-[1-\{g(u)\}^2]^{\frac{1}{2}}[1-\{g(v)\}^2]^{\frac{1}{2}}$

the author easily deduces that

(4) $g(u+v)+g(u-v)=2g(u)g(v)$.

Kaczmarz [Fund. Math. **6** (1924), 122–129] has proved that the only measurable solution of (4) is given by $g(u)=\cos(ku)$, $k=$const. This therefore solves (3), and on writing $g(u)=x$, $ku=$ arc cos $x=f(x)$ in (3) the author then obtains the solution of (1) given by Aczél and Varga. A similar method applies to (2). *Charles Fox.*

See also: Functions of Real Variables: de Rham. **Ordinary Differential Equations:** Koval. **Probability:** Aczél and Egerváry.

Integral and Integrodifferential Equations

Sakhnovich, L. A. The spectral analysis of Volterra operators and some inverse problems. Dokl. Akad. Nauk SSSR (N.S.) **115** (1957), 666–669. (Russian)

Let K be a bounded operator on $H=L_2[0, m]$, given by $(Kf)(x)=\int_0^x K(x, t)f(t)dt$. A sufficient condition is given that K should be equivalent to the operator J^2 which sends f into $\int_0^x (x-t)f(t)dt$. The condition is that: 1) there exist the bounded derivatives $(x\geqq t)$

$$\frac{\partial^{j+k}}{\partial x^j\partial t^k}K(x, t)\ (j, k=0, 1, 2), \frac{d}{dx}\left[\frac{\partial^2}{\partial x^2}K(x, t)_{t=x}\right];$$

2) $K(x, x)=0$; and 3) $|\partial K(x, t)/\partial x_{t=x}|>0$.

R. Arens (Los Angeles, Calif.).

Yoshimatsu, Senjiro. Sur le théorème dans les équations intégrales. Mem. Osaka Univ. Lib. Arts Ed. Ser. B. **5** (1956), 1–4.

The note extends the theory of Erhard Schmidt on nonsymmetric kernels in integral equations [Math. Ann. **63** (1907), 433–476, pp. 461–466] to a Hilbert space setting and obtains the results one expects. If A is a completely continuous linear transformation on the space H to H, then AA^* and A^*A are symmetric, have the same positive characteristic values λ_i, and characteristic elements φ_i, ψ_i, respectively, related: $\varphi_i=\lambda_i^{\frac{1}{2}}A\psi_i$, $\psi_i=\lambda_i^{\frac{1}{2}}A^*\varphi_i$. For x of H, we have $Ax=\sum_i (x, \varphi_i)\varphi_i/\lambda_i^{\frac{1}{2}}$. Conditions for the solution of the equation $Ax=y$ are immediate. *T. H. Hildebrandt.*

Korobeinik, Yu. F. A solution of a mixed problem by the Fourier method for a certain integro-differential equation. Dokl. Akad. Nauk SSSR (N.S.) **114** (1957), 14–17. (Russian)

This note discusses conditions for uniqueness and existence of solutions of a linear integro-differential equation.

M. M. Day (Urbana, Ill.).

Albrycht, Jerzy. On certain systems of non-linear integral and integro-differential equations. Zeszyty Nauk. Uniw. Mickiewicza. Mat.-Chem. **1** (1957), 19–23. (Polish. Russian and English summaries)

Theorems 1 and 3 are extensions of Theorems 1 and 2 of O. Ženhèn in Dokl. Akad. Nauk SSSR (N.S.) **86** (1952), 229–230 [MR **14**, 383]. Let y be an n-dimensional vector, x a real variable; then $y=f(x, y, \int_a^b k(x, t, y(t))dt)$ has a unique solution, and $y^{(n)}=f(x, y, y', y'', \cdots, y^{(n-1)}, \int_a^b k(x, t, y, y', \cdots, y^{(m)})dt)$, where $y^{(k)}$ is the kth derivative of y with respect to x, $m<n$, has a unique solution which satisfies given initial conditions and is valid in the neighborhood of a point defined by the initial conditions, if f and k satisfy Lipschitz conditions with suitable restrictions on the corresponding constants. The conditions and the proof are direct extensions of the 1-dimensional case considered by Ženhèn. Theorems 2 and 4 are analogous extensions of the Theorems 1 and 2 of Ženhèn in Dokl. Akad. Nauk SSSR (N.S.) **91** (1953), 1261–1262 [MR **15**, 435]. In these theorems Lipschitz conditions are replaced by uniform continuity, boundedness and measurability conditions, respectively. Then the existence of at least one solution follows. *C. Masaitis.*

See also: Integral Transforms: Jaeckel. **Fluid Mechanics, Acoustics:** Lamb.

Calculus of Variations

★ **Sigalov, A. G. On the correctness of formulation of two-dimensional problems of the calculus of variations.** Pamyati Aleksandra Aleksandroviča Andronova [In memory of Aleksandr Aleksandrovič Andronov], pp. 535–540. Izdat. Akad. Nauk SSSR, Moscow, 1955. 36.40 rubles.

Consider the minimizing surfaces T_n for quasiregular positive definite double integral problems of the calculus of variations, corresponding to Jordan boundary curves G_n. The existence of these surfaces was proved some years back by the author [Dokl. Akad. Nauk SSSR (N.S.) **70** (1950), 769–772; **71** (1950), 617–620; MR **11**, 603, 604] and the reviewer [Thesis, Univ. of California, 1949], and slightly later, independently, by L. Cesari. Suppose that

$G_n \to G_0$ in the sense of Fréchet, and that they have bounded length. Suppose further that, for some $\varepsilon > 0$, all except finitely many of the T_n are within an ε-neighborhood U_ε (in the Fréchet metric) of the surface T_0 corresponding to G_0. Suppose finally that there is no other surface T minimizing the given integral for G_0 and lying in U_ε. Then the surfaces T_n converge to T_0 in the sense of Fréchet and the values of the integral on the T_n converge to the value on T_0. Otherwise put, in the neighborhood of an isolated solution for a given boundary curve, the solutions depend continuously on the initial data. In particular this is the case if the solution for G_0 is unique.

J. M. Danskin (Princeton, N.J.).

Lepage, Th. H. Sur certains opérateurs différentiels. Acad. Roy. Belgique. Bull. Cl. Sci. (5) **39** (1953), 143–155.

Gestützt auf den Begriff einer "differentiablen" Differentialform [P. Gillis derselbe Bull. (5) **29** (1943), 175–186; MR **7**, 120] und Gebrauch machend von früheren Ergebnisse des Verfassers [ibid. (5) **38** (1952), 412–415; MR **14**, 126] werden einige mit einer festen quadratischen Differentialform $\Gamma = \sum_1^n dx_i \wedge dp_i$ verknüpfte Differentialoperationen definiert und zur Ermittlung der ersten und zweiten Variation des Integrals

$$\int L\left(x, \frac{\partial u}{\partial x_i}, \frac{\partial^2 u}{\partial x_i \partial x_j}\right) dx_1, \cdots, dx_n$$

verwendet. *R. W. Weitzenböck* (Zbl **51** (1954), 80).

See also: Measure, Integration: Cesari and Fullerton.

TOPOLOGICAL ALGEBRAIC STRUCTURES

Topological Groups

★**Pontrjagin, L. Topological groups.** Translated from the Russian by Emma Lehmer. Princeton University Press, Princeton, N.J., 1939. (Fifth printing, 1958). ix+299 pp. $2.75.

This is a paperbound reprint of the book reviewed in MR **1**, 44.

Schöneborn, Heinz. Bemerkungen zur primären Zerlegung torsionstopologischer, abelscher Gruppen. Arch. Math. **8** (1957), 23–29.

Torsion topological groups are abelian topological groups with a topology defined by open subgroups such that for every element g, the multiples ng have 0 as an accumulation point. Furthermore the author postulates ordinary completeness and dual completeness of the group A (i.e., if for a subgroup A' and every compact subgroup K, $A' \cap K$ is open in K, then A' is open in A). Such groups A can be split into their primary components $A^{(p)}$ in the sense of unique infinite sums $\sum a^{(p)}$ such that the $a^{(p)}$ generate a compact subgroup. A uniqueness theorem is given, and the related primary decompositions of subgroups, factor groups and character groups are studied. *H. Freudenthal* (Utrecht).

See also: Topological Vector Spaces: Fischer. **Banach Spaces, Banach Algebras, Hilbert Spaces:** Naĭmark.

Lie Groups and Algebras

Kojima, Jun. On the Pontrjagin product mod 2 of spinor groups. Mem. Fac. Sci. Kyusyu Univ. Ser. A. Math. **11** (1957), 1–14.

The ring $H_*(\mathrm{Spin}(n), Z_2)$ is explicitly determined in the cases $n = 2^s+1$, $n = 2^s+2$, and partly in the cases $n = 2^s+i$, $3 \leq i \leq 2^s$, $s \geq 3$. In the case $n = 2^s+1$, $s \geq 3$, H_* is found to be commutative. It is non-commutative in the other cases ($n \geq 10$). In general outline the method employed is the same as the one used by Borel in the calculation of $H_*(\mathrm{Spin}(n), Z_2)$, $n \leq 10$. *W. T. van Est* (Utrecht).

Berezin, F. A. Laplace operators on semi-simple Lie groups. Trudy Moskov. Mat. Obšč. **6** (1957), 371–463. (Russian)

This is a detailed exposition with proofs of the results announced before [Dokl. Akad. Nauk SSSR (N.S.) **107** (1956), 9–12; **110** (1956), 897–900; MR **17**, 1109; **19**, 292]. *W. T. van Est* (Utrecht).

Gurevič, G. B. Conditions of isomorphism of standard nilalgebras. Trudy Moskov. Mat. Obšč. **6** (1957), 165–193. (Russian)

The purpose of this paper is to find isomorphism conditions in terms of the characteristics of the standard nilalgebras involved [see Gurevič, Mat. Sb. N.S. **35(77)** (1954), 437–460; MR **17**, 509]. To this end, first a rule is found to compute the order of the nilalgebra from its characteristic. The order is the least exponent m such that $(\mathrm{ad}\, x)^m = 0$ for every x from the nilalgebra. Now it turns out that two standard nilalgebras of order ≥ 3 are isomorphic if and only if they have either the same or dual characteristics; the latter means that the two linear nilalgebras are contragredient representations of the same Lie algebra. When the order is 2 or 1 the isomorphism conditions are more complicated but still expressible in terms of the characteristic. *W. T. van Est* (Utrecht).

Topological Vector Spaces

Koecher, Max. Positivitätsbereiche im R^n. Amer. J. Math. **79** (1957), 575–596.

The author considers the real field R with the natural topology and a vector space X over R. The norm, $|x|$, of $x \in X$ is defined to have the usual properties. A mapping $L(a, b)$ of $X \times X$ into R is called a bilinear form of X if $L(a, b) = L(b, a)$ for all a and b in X and if, for a fixed a, $L(a, x)$ is linear and continuous. Denote by $X(L)$ the set of a in X for which $L(a, x) = 0$ for all x in X. If L is a bilinear form of X, then a subset Y of X is called a "isoptivity region" (Positivitätsbereich) of X with characteristic L if: 1) Y is open and not empty; 2) for all a and b in Y, $L(a, b) > 0$; and 3) for x not in Y, there is an a in the complement of Y which does not belong to $X(L)$ and for which $L(a, x) \leq 0$. Let X be n-dimensional. For a non-degeneratie positivity region Y, $\Sigma(Y)$ denotes the multiplicative group of non-singular real n-rowed matrices W with $W\bar{Y} = \bar{Y}$, where $\bar{Y}$ is the topological closure of Y. Then a norm of Y is a function $f(y)$ defined on Y which has the following properties: $f(y)$ is a continuous and po-

sitive for all y in Y, and $f(Wy)=\|W\|f(y)$ for all W in $\Sigma(Y)$, where $\|Y\|$ denotes the absolute value of the determinant of W. A positivity region Y is called homogeneous if: 1) $L(a, x)=0$ only if $a=0$; 2) for any two points a and b in Y, there is an automorphism W of Y such that $a=Wb$.

Two important results proved are: Th. 5: If Y_1 and Y_2 are two homogeneous positivity regions with norms $N_1(y)$ and $N_2(y)$, if Y_1 and Y_2 are not disjoint and if $N_1(y)=N_2(y)$ for all y in both Y_1 and Y_2, then $Y_1=Y_2$. Th. 6: If W is a real non-degenerate matrix, it is an automorphism of Y if and only if $WY\cap Y\neq\emptyset$ and $|W|\cdot N(W^{-1}y)=N(y)$ for all y in $WY\cap Y$. *B. W. Jones.*

Shimogaki, Tetsuya. On the norms by uniformly finite modulars. Proc. Japan Acad. **33** (1957), 304–309.

A norm on a universally continuous normed semi-ordered linear space R [see H. Nakano, Modulared semi-ordered linear spaces, Maruzen, Tokyo, 1950; MR **12**, 420] is said to be finitely monotone, if for every $\gamma>0$ there exists an integer n such that $\|x_i\|\geqq 1$, $|x_i|\cap|x_j|=0$ for every $i\neq j$ imply $\|\sum_{i=1}^{n}x_i\|>\gamma$. The author shows that spaces R exist in which the norm is monotone complete and continuous but not finitely monotone, and that in such spaces no modular can be defined to be topologically equivalent to this norm. *I. G. Amemiya* (Kingston, Ont.).

Shimogaki, Tetsuya. Note on Orlicz-Birnbaum-Amemiya's theorem. Proc. Japan Acad. **33** (1957), 310–313.

The author reverifies the following theorem: if R is a non-atomic modulared semi-ordered linear space with a monotone complete finite modular m, then for every $\varepsilon>0$ there exists $\gamma>0$ such that $m(x)\geqq\varepsilon$ implies $m(2x)\leqq\gamma m(x)$. [See Amemiya, J. Fac. Sci. Hokkaido Univ. Ser. I. **13** (1956), 60–64; MR **18**, 491.]

He also shows that a monotone complete modular m of a discrete (atomic) space R is finite if and only if: $m(x)<+\infty$ for every atomic element x, and there exist $\varepsilon'>\varepsilon>0$ and $\gamma>0$ such that $\varepsilon\leqq m(x)\leqq\varepsilon'$ implies $m(2x)\leqq\gamma m(x)$.

These two theorems are generalizations of the theorems of Orlicz and Birnbaum [cf. the paper cited above].

I. G. Amemiya (Kingston, Ont.).

Kaplan, Samuel. On the second dual of the space of continuous functions. Trans. Amer. Math. Soc. **86** (1957), 70–90.

Let C denote the Banach space of real continuous functions on a compact Hausdorff space X, let L denote the dual of C and let M denote the dual of L. These three spaces may also be considered as vector lattices. The author makes use of this fact to investigate properties of M which involve order, in particular those related to the usual integration processes.

The space of bounded functions on X is identified with a subspace M_0 of M, and M is a topological direct sum of M_0 and a second subspace M_1. For every bounded function f on X the equivalence class of f in M under the homomorphism $M\to M_0$ is shown to contain two distinguished elements f^* and f_* such that $f^*(\mu)$ and $f_*(\mu)$ are respectively the superior and inferior integrals with respect to μ, for any Radon measure μ in L. If f is integrable with respect to every μ in L, $f^*=f_*$ and their common value at μ is $\int f\,d\mu$. The elements f^* and f_* of M are defined for every f in M by methods paralleling those used in standard integration procedure.

The set U consisting of elements f of M such that $f=f^*=f_*$ is shown to be a linear subspace of M closed under the norm and also closed under countable unions and intersections. Order-convergence in M is examined, and U is shown to consist precisely of the elements of M which are, under order-convergence, limits of directed systems of continuous functions (embedded in M in the usual way). Under the topology $|w|(M, L)$ defined on M by the family of semi-norms $\|f\|_\mu=|f|(\mu)$, $(\mu\in L_+)$, C is shown to be dense in M.

In the final section a beginning is made in the development of integration theory in the context of M. For a given $\mu\in L_+$, the classes of μ-negligible and μ-integrable elements of M are defined, their properties are examined, and the analogue of the Lebesgue bounded convergence theorem is proved. *W. R. Transue* (Gambier, Ohio).

Day, Mahlon M. Every, l-space is isomorphic to a strictly convex space. Proc. Amer. Math. Soc. **8** (1957), 415–417.

The author presents a proof of the result stated in the title. He points out that this example does not settle the possibility that all smooth spaces might be strictly convex.

J. P. LaSalle (Notre Dame, Ind.).

Shibata, Toshio. On a family of subspaces of the space $(\mathcal{E})$. Sci. Rep. Tokyo Kyoiku Daigaku. Sect. A. **5** (1956), 237–259.

Let X be a euclidean space and W a family of continuous functions on X. The author defines a space A of C^∞ functions on X as follows: $f\in A$ if for each constant coefficient differential operator D on X and each $w\in W$, wDf is bounded on X. The topology of A is defined by means of the semi-norms $\max|w(x)Df(x)|$. In the case that W consists of the functions $(1+|x|)^k$ $(k=1, 2, \cdots)$, the A becomes the space S of L. Schwartz.

Under additional assumptions on W, the author studies the dual A' of A, the space of C^∞ functions g such that $gf\in A$ for every $f\in A$, and he also studies certain properties of convolution which are related to these spaces.

L. Ehrenpreis (Waltham, Mass.).

Schwartz, Laurent. Distributions semi-régulières et changements de coordonnées. J. Math. Pures Appl. (9) **36** (1957), 109–127.

Une distribution $T\in\mathcal{D}'_{x,y}(x\in R^m, y\in R^n)$ est dite semi-régulière en x, si elle s'identifie à une fonction indéfiniment différentiable de x à valeurs dans $\mathcal{D}'_y$; c'est-à-dire si $T\in\mathcal{E}_x(\mathcal{D}'_y)\cong\mathcal{E}_x\hat{\otimes}\mathcal{D}'_y$ [voir Séminaire Schwartz Fac. Sci. Paris, 1953/1954; MR **17**, 764]. Le but essentiel de ce travail est de chercher sous quelles conditions la notion de distribution semi-régulière est indépendante du système de coordonnées choisi. A cet effet l'auteur étudie la structure des distributions $T\in(\mathcal{E}_x\hat{\otimes}\mathcal{D}'_y)'$: ce sont les distributions T de la forme

$$T=\sum_{|p|\leq k}D_x^p g_p(\hat{x}, \hat{y}),$$

où les g_p sont des fonctions de x, y, à support compact, indéfiniment dérivables en y et dont toutes les dérivées en y sont des fonctions continues de x, y.

Une distribution $T\in\mathcal{D}'_{x,y}$ est dite intégralement semiré-gulière en y, si pour toute $\alpha\in\mathcal{D}_x$, et toute $\beta\in\mathcal{D}_y$ on a $\alpha\beta T\in(\mathcal{E}_x\hat{\otimes}\mathcal{D}'_y)'$; cela entraîne que T est semi-régulière en y. Ces deux notions se généralisent aisément au cas des courants sur une variété indéfiniment différentiable V_{m+n}, où l'on prend un système de "coordonnées globales"

$x_1, \cdots, x_m, y_1, \cdots, y_n$. On a enfin ce résultat fondamental: si l'on se donne sur V^{m+n} deux systèmes de coordonnées (x, y) et (ξ, η), vérifiant certaines conditions (assez larges), alors tout courant intégralement semi-régulier en x dans le système (x, y) est semi-régulier en ξ dans le système (ξ, η). *J. Sebastião e Silva* (Lisbonne).

Ringrose, J. R. Precompact linear operators in locally convex spaces. Proc. Cambridge Philos. Soc. **53** (1957), 581–591.

Let E be a locally convex Hausdorff linear space over the complex field, and E' its dual, with the strong topology (uniform convergence on the bounded subsets of E). The author considers the relation between the compactness or precompactness of a linear operator T from E to itself, and the corresponding properties of the adjoint operator T'. It is proved that if T is precompact, and A is a bounded linear operator from E to itself, then $T'A'$ is compact; in particular, $(T')^2$ is compact. The basic idea of the proof is the reduction of the problem to one on operators in a Banach space, by completing a suitable quotient space of E. As the author remarks, very similar considerations have been employed by Altman [Studia Math. **13** (1953), 194–207; MR **15**, 436]. An example is given in which T is compact but T' is not precompact.

Spectral properties of T and T' are investigated, when T is compact, and the expected results are obtained. Some of these had previously been obtained otherwise by Leray [Acta Sci. Math. Szeged **12** (1950), Pars B, 177-186; MR **12**, 32] and the reviewer [J. London Math. Soc. **29** (1954), 149–156; MR **15**, 801]. The paper concludes with a discussion of the spectral properties of a precompact but not necessarily compact operator, and the pathological conditions which may occur.

Reference should be made to closely related work by Köthe [Portugal. Math. **13** (1954), 97–104; MR **16**, 715]. *J. H. Williamson* (Belfast).

Fischer, H. R. Differentialkalkül für nicht-metrische Strukturen. Ann. Acad. Sci. Fenn. Ser. A. I. no. **247** (1957), 15 pp.

Generalising the notion of Fréchet derivative from normed vector spaces to topological groups and locally convex linear spaces, the author is able to find new forms of the standard basic theorems on differentiation. *J. Schwartz* (Upton, N.Y.).

See also: Measure, Integration: Cotlar and Ricabarra; Sawashima. **Banach Spaces, Banach Algebras, Hilbert Spaces:** Naĭmark.

Banach Spaces, Banach Algebras, Hilbert Spaces

Weston, J. D. A characterization of separable Banach spaces. J. London Math. Soc. **32** (1957), 186–187.

A Banach space is shown to be separable if and only if it is the completion of the range of a completely continuous linear transformation. The bridge between the two conditions is provided by the presence, in either case, of a compact fundamental set in the Banach space. *M. Jerison* (Lafayette, Ind.).

Weiss, Guido. An interpolation theorem for sublinear operators on H_p spaces. Proc. Amer. Math. Soc. **8** (1957), 92–99.

In this paper, a generalization of the Riesz convexity theorem given by Calderón and Zygmund [Amer. J. Math. **78** (1956), 282–288; MR **18**, 586] for operators defined between L_p spaces is further generalized to operators from H_p spaces to L_p spaces. An operator $T: X \to Y$ between two normed linear spaces is sublinear if $T(F_1+F_2)$, $T(kF_1)$ are defined uniquely whenever TF_1, TF_2 are defined, and

$$\|T(F_1+F_2)\| \leqq \|TF_1\| + \|TF_2\|, \quad \|T(kF)\| = |k| \cdot \|TF\|.$$

If the domain of definition of T includes H_p, $p>0$, and $T(H_p)$ is included in $L_q(M, \mu)$, $q \geqq 1$, (M, μ) some measure space, then T is of type (p, q) if there exists a constant M such that $\|TF\|_q \leqq M\|F\|_p$ for all $F \in H_p$, where $\| \ \|_q$ is the L_q norm, and $\| \ \|_p$ the H_p norm. The author's theorem then states that if T is sublinear of types $(1/\alpha_1, 1/\beta_1)$ and $(1/\alpha_2, 1/\beta_2)$ simultaneously with constants M_1, M_2 respectively, $1/\alpha_i > 0$, $1/\beta_i \geqq 1$, $(i=1, 2)$, then if $\alpha = (1-t_0)\alpha_1 + t_0\alpha_2$, $\beta = (1-t_0)\beta_1 + t_0\beta_2$, $0 \leqq t_0 \leqq 1$, there exists an absolute constant K such that $\|TF\|_{1/\beta} \leqq KM^{(1-t_0)}M_2^{t_0}\|F\|_{1/\alpha}$. *R. E. Fullerton.*

Wolf, František. Operators in Banach space which admit a generalized spectral decomposition. Nederl. Akad. Wetensch. Proc. Ser. A. **60**=Indag. Math. **19** (1957), 302–311.

Let B be a Banach space, E a function from the interval (a, b) to B which is continuous except possibly at finitely many points. By $C^n(a, b)$ we mean the class of functions possessing n continuous derivatives on (a, b). For f in $C^n(a, b)$ the author defines the element $\int_a^b f(\lambda)d^nE(\lambda)$ in B as equal to

$$(-1)^n \int_a^b f^{(n)}(\lambda)E(\lambda)d\lambda + \sum_{l=0}^{n-1} (-1)^l [f^l(b)E_{n-l-1}^+ - f^l(a)E_{n-l-1}^-],$$

where the E_i^+ and E_i^- are fixed vectors in B. Denote by $\mathfrak{A}_k$ the class of operators A on a given Banach space X such that A has spectrum on the unit circle and such that there exists a function E from $(0, 2\pi)$ to the space of bounded operators on X with

$$f(A) = \int_0^{2\pi} f(e^{i\theta})d^kE(\theta)$$

whenever $f(\lambda) = \sum_{-p}^{p} c_n\lambda^n$. The right-hand side of this formula is then defined for every f in C^k and is again denoted by $f(A)$. The class $\mathfrak{A}_0$ is closely related to Dunford's spectral operators [Pacific J. Math. **2** (1952), 559–614; MR **14**, 479]. The author now introduces two classes of operators with spectrum on $|\lambda|=1$: $\mathfrak{B}_k$ consists of all A with $\|R_\lambda\| = O(1-|\lambda|)^{-k}$, where R_λ is the resolvent $(\lambda I - A)^{-1}$, and $\mathfrak{C}_k$ consists of all A with $\|A^n\| = O(|n|^k)$ as $n \to \pm\infty$. Theorem: $\mathfrak{B}_k \subseteqq \mathfrak{A}_l$ and $\mathfrak{C}_k \subseteqq \mathfrak{A}_l$ for $l > k+1$. Also $\mathfrak{A}_k \subseteqq \mathfrak{C}_{k+1}$ and $\mathfrak{A}_k \subseteqq \mathfrak{C}_{k+1}$. The last half of the paper is concerned with certain kinds of invariant subspaces of the operators A in $\mathfrak{A}_n$. Fix a closed subset S of the spectrum of A and choose a function φ_S in C^n with $S = \{\lambda | \varphi_S(\lambda) = 0\}$. Form the operator $\varphi_S(A)$ and put $\mathfrak{M}(S) =$ null-space of $\varphi_S(A)$. Then $\mathfrak{M}(S)$ is an invariant subspace of A depending only on S. For each x in X, let σ_x be the set of singularities of $R_\lambda x$ as λ varies. Theorem: $\mathfrak{M}(S) = \{x | \sigma_x \subseteqq S\}$. The author next discusses a formulation of a theorem of the reviewer [Duke Math. J. **19** (1952), 615–662, Th. 1; MR **14**, 384] which asserts that, for operators A in a more extensive class than any $\mathfrak{A}_k$, $\{x | \sigma_x \subseteqq S\}$ is a closed invariant subspace. Finally, a way of associating invariant subspaces to open subsets is considered. *J. Wermer* (Providence, R.I.).

Dinculeanu, Nicolae. Sur la représentation intégrale de certaines opérations linéaires. C. R. Acad. Sci. Paris **245** (1957), 1203–1205.

For each z in the locally compact space Z let $E(z)$ be a Banach space. A fundamental family $\mathscr{A}$ in the cartesian product of the spaces $E(z)$, $z \in Z$, is a linear subspace such that the map $z \to |x(z)|$ is continuous for each x and such that for each z in Z the set $\{x(z) | x \in \mathscr{A}\}$ is dense in $E(z)$. It is supposed that the family $\mathscr{A}$ satisfies the axiom of Godement: There is a denumerable part $\mathscr{A}_0$ of $\mathscr{A}$ such that for every z in Z the set $\{x(z) | x \in \mathscr{A}_0\}$ is dense in $E(z)$. Let F be a separable Banach space which is the dual of the Banach space B. Let $G(z)$ be the space of linear bounded maps from $E(z)$ into F, let $\mathscr{G}$ be the family $(G(z), z \in Z)$ and $\mathscr{C}(\mathscr{G})$ the cartesian product of the elements of $\mathscr{G}$. It is shown that every continuous linear map f of $L^1_{\mathscr{A}}(\mu)$ into F has the form $f(x) = \int U_f(z)x(z)\mu(dz)$ where $U_f \in \mathscr{C}(\mathscr{G})$ and that $|f| = \mu_f - \text{ess sup}_{z \in Z} |U_f z|$. A corollary concerns the case where Z is compact and the map f is defined on the set of those elements of the cartesian product $E(z)$, $z \in Z$, which are continuous with respect to the family $\mathscr{A}$. *N. Dunford.*

Schatz, J. A. Representation of Banach algebras with an involution. Canad. J. Math. **9** (1957), 435–442.

The author proves that a complex Banach algebra with an involution satisfying $\|a^*\| = \|a\|$ can be mapped onto an algebra of operators over a Banach space. The involution in the operator algebra is defined as adjointness with respect to a skew-symmetric, continuous nondegenerate bilinear functional. The isomorphism is nondecreasing in general, norm-preserving for selfadjoint elements. In case the involution satisfies $\|a^*a\| = \|a\|^2$, the author rederives, with the aid of a theorem of the reviewer [Comm. Pure Appl. Math. **7** (1954), 633–647; MR **16**, 832], the Gel'fand-Naĭmark theorem.

P. D. Lax (New York, N.Y.).

Wolfson, Kenneth G. A note on the algebra of bounded functions. II. Proc. Amer. Math. Soc. **7** (1956), 852–855.

The problem of characterizing those B^*-algebras B with identity which are isomorphic (in a norm- and *-preserving manner) to an algebra of all bounded complex-valued functions on a discrete space S has been treated by the author [Proc. Amer. Math. Soc. **5** (1954), 10–14; MR **15**, 633] and subsequently by Heider [ibid. **6** (1955), 305–308; MR **16**, 935]. The author shows that the characterizations of Heider can be derived from his earlier work. He also shows that B is an algebra of the above type if and only if (1) the space M of maximal ideals contains a dense set of isolated points and (2) every regular open set in M is closed. *B. Yood* (Eugene, Ore.).

★ **Наймарк, М. А. [Naĭmark, M. A.] Нормированные кольца. [Normed rings.]** Gosudarstv. Izdat. Tehn.-Teor. Lit., Moscow, 1956. 487 pp. 20.40 rubles.

This treatise is not merely a technical report on Banach algebras ("normed rings" in Soviet parlance), but is actually a compendium of functional analysis, containing a full treatment of those parts of the subject that are relevant to the theory of normed algebras. In theory, it is accessible to anyone who can read Russian, knows some algebra, and is familiar with elementary analysis. The scope of the book is enormous: starting with the simplest concepts, the author finishes with a report on recent developments in the theory of W^*-algebras.

Chapter I, titled "Basic information about topology and functional analysis", occupies about 3/10 of the book. It contains a standard discussion of linear spaces, topological spaces, topological linear spaces, normed linear spaces, and Hilbert spaces. The Hahn-Banach theorem for complex linear spaces is attributed to Suhomlinov [Mat. Sb. N.S. **3(45)** (1938), 353–358], although the independent and practically simultaneous publication by Bohnenblust and Sobczyk [Bull. Amer. Math. Soc. **44** (1938), 91–93] is also mentioned. This is the general pattern for bibliographic reference throughout the book. Convex functionals, the Kreĭn-Mil'man theorem, etc., are treated very adequately. There is an excellent introductory discussion of linear operators in Hilbert space. The chapter closes with a treatment of integration on a locally compact Hausdorff space T, containing several interesting innovations. Let L be the space of all complex-valued continuous functions on T that vanish in neighborhoods of infinity, and let I be a complex linear functional on L that is non-negative for non-negative real functions in L. {The proof that $|I(x)| \leqq I(|x|)$ is needlessly long; the following short proof is due to K. Itô. For $x \in L$, write $I(x) = \rho e^{i\phi}$. Then $|I(x)| = \rho = e^{-i\phi} I(x) = I(e^{-i\phi}x) = I(y_1) + iI(y_2) = I(y_1) \leqq I(|y_1|) \leqq I(|x|)$.} The functional I is extended by the Daniell method, and a subadditive upper integral $\bar{I}$ is obtained for all non-negative functions on T. The space $\mathfrak{L}_1$ is defined as the closure of L in the normed space of all complex functions x for which $\bar{I}(|x|)$ is finite. Measure and measurable sets are introduced in such a way as to avoid the difficulties arising if T is not σ-compact (a set $A \subset T$ is measurable if, for every compact subset B of T, the characteristic function of $A \cap B$ is in $\mathfrak{L}_1$). It is shown that the original functional I is the Lebesgue integral with respect to the measure obtained from $\bar{I}$. The Fubini and Radon-Nikodym theorems are proved. This section provides the clearest development known to the reviewer of its subject.

Chapter II, titled "Basic concepts and asertions of the theory of normed rings", begins with an elementary discussion of algebras over the complex field. The Jacobson radical (here called simply the radical), adjunction of a unit, and so on, are taken up. Proposition VI on page 150 is false as stated. It asserts that if R is an algebra without unit and R' is R with a unit adjoined, then the mapping $I' \to I' \cap R$ is a one-to-one mapping of the family of right ideals I' of R' not contained in R onto the family of all regular right ideals of R. The case in which R is all 2×2 complex matrices with 2nd row zero provides a counter-example. {The theorem is true if right ideals are replaced by maximal right ideals.} The Gel'fand-Mazur theorem is given in the following form: if R is a topological field over the complex numbers in which inversion is continuous, then R is the complex numbers. Normed algebras and Banach algebras are defined and elementary properties are proved. The chapter closes with a discussion of algebras over the complex numbers admitting an involution $x \to x^*$ (such algebras are called here symmetric) and of positive functionals on such algebras. {Proposition IV on page 172 is false without the added hypothesis that $f(x) = \overline{f(x^*)}$.}

Chapter III, titled "Commutative normed rings", contains mostly standard material. For a commutative Banach algebra R with unit, the space $\mathfrak{M}$ of maximal ideals is constructed, topologized by the weak *-topology, and identified with the set of all multiplicative linear functionals on R. Analytic functions of one or more ele-

ments of R are studied. The Šilov boundary is discussed. It is proved that if M is a maximal ideal in the boundary of a commutative Banach algebra R with unit, then M can be extended to a maximal ideal in any commutative Banach algebra with unit that contains R. The author calls a symmetric commutative Banach algebra with unit completely symmetric if $x^*(M)=\overline{x(M)}$ for all $x \in R$ and all maximal ideals M in R. This is equivalent to the condition that $(e+xx^*)^{-1}$ exist for all $x \in R$. The standard representation theorems for completely symmetric algebras as dense subalgebras of $\mathfrak{C}(\mathfrak{M})$ are proved. Commutative Banach algebras that are regular in Šilov's sense are next studied. {Proposition I on page 199 seems to be incorrect unless the space in question is normal.} The discussion of regular Banach algebras culminates in Šilov's theorem, which gives conditions under which an element in the kernel of the hull of an ideal lies in the ideal. The chapter concludes with a discussion of completely regular symmetric normed algebras, which are defined as those in which $\|xx^*\|=\|x\|^2$. It is shown that every completely regular commutative Banach algebra with unit is isomorphic with $\mathfrak{C}(\mathfrak{M})$.

Chapter IV is titled "Representations of symmetric rings". A representation of a symmetric algebra R is a *-preserving homomorphisms $x \to T_x$ of R into the algebra $\mathfrak{B}(\mathfrak{H})$ of bounded operators on a Hilbert space $\mathfrak{H}$. A cyclic representation is defined as one such that for some $\xi_0 \in \mathfrak{H}$ the set $\{T_x\xi_0\}_{x \in R}$ is dense in $\mathfrak{H}$. {To avoid troublesome special cases when there is no unit, it would be convenient to require instead that $\{\xi_0\} \cup \{T_x\xi_0\}_{x \in R}$ be dense in $\mathfrak{H}$.} The one-to-one correspondence between cyclic representations of R and positive functionals on R is set up. All cyclic representations of an algebra $\mathfrak{C}(X)$ (X a compact Hausdorff space) are found (they are simply multiplication in L_2 for a certain Radon measure μ on X). The spectral theorem is obtained from this, by an argument that could be slightly shortened. The possibility of and methods for imbedding a symmetric algebra in an algebra of operators are carefully explored. For this purpose, regular norms are introduced; a norm on a symmetric algebra R with unit is called regular if every positive functional on R can be extended to a positive functional on the completion of R; i.e., $|f(x)| \leq f(e)\|x\|$ for all $x \in R$ and every positive functional f on R. If R admits a regular norm, then it admits a smallest one, which is given by the expression $\sup(f(x^*x))^{\frac{1}{2}}$, where the supremum is taken over all positive functionals f on R such that $f(e)=1$. The connection between irreducible representations of the symmetric algebra R and indecomposable positive functionals is described in detail. Abstract Herglotz-Bochner theorems for symmetric and completely symmetric commutative algebras, with and without unit, are obtained (representation of positive functionals as integrals over the maximal ideal space). The abstract Plancherel theorem, dealing with extensibility of positive functionals from dense ideals, and their representation by not necessarily bounded integrals on the maximal ideal space, is proved for a completely symmetric commutative algebra without unit. The chapter also contains the author's generalization of Schur's lemma and a discussion of representations of $\mathfrak{B}(\mathfrak{H})$.

Chapter V is titled "Certain special rings". Not necessarily commutative completely symmetric algebras R are dealt with first. (The property that $(e+xx^*)^{-1}$ exists for all $x \in R$ is taken as the definition.) Let R_1 be a closed symmetric subalgebra of R containing e. Then every continuous symmetric (irreducible) representation of R_1 can be extended to a similar (irreducible) representation of R. A symmetric algebra R with unit is completely symmetric if and only if

$$\sup f(x^*x)=\lim_{n \to \infty} (\|(xx^*)^n\|^{1/n}),$$

the supremum being taken over all positive functionals f such that $f(e)=1$. It is next proved that every complete completely regular algebra R with unit is completely symmetric. [For other approaches to this theorem, see the review of Fukamiya, Kumamoto J. Sci. Ser. A. **1** (1952), no. 1, 17–22; MR **14**, 884; **15**, 1139]. The discussion ends with the proof that every completely regular symmetric algebra is isomorphic to an algebra of operators in some Hilbert space. Dual rings are discussed, as is their application to H^*-algebras (here called Hilbert rings). The structure of H^*-algebras is completely described. The chapter closes with a discussion of algebras of vector functions. Given a topological space T and a Banach algebra R_t, for every $t \in T$, let $\mathfrak{C}(T, R_t)$ be the subset of the Cartesian product of all R_t such that $\|x(t)\|$ is a bounded continuous function on T. $\mathfrak{C}(T, R_t)$ is a Banach algebra with pointwise operations and the obvious norm. Closed subalgebras of $\mathfrak{C}(T, R_t)$, here called rings of vector functions, are analyzed in some detail.

Chapter VI is titled "Group rings". Topics treated are: groups as such; topological groups; the Haar integral; the algebra $L_1(G)$; positive functionals on $L_1(G)$; the Gel'fand-Raĭkov theorem; Fourier transforms for a locally compact commutative group; arbitrary unitary representations of a locally compact commutative group; the Tauberian theorem; unitary representations of compact groups; tensor products of representations; and the Tannaka-Kreĭn duality theorem. The treatment appears nearly flawless.

Chapter VII is titled "Rings of operators in Hilbert space". Von Neumann's and Murray-von Neumann's work on weakly closed self-adjoint algebras of operators is treated with great clarity and elegance.

Chapter VIII, titled "Decomposition of a ring of operators into irreducible rings", is of course incomplete, since the subject is far from closed as the present time. J. M. G. Fell has pointed out to the reviewer that the proof of Proposition VII, p. 458, is incomplete.

The book concludes with a huge bibliography, in which 414 items are listed. The book is enlivened throughout by a large number of concrete examples of algebras, representations, and the like. These increase its interest considerably. In addition to the venial errors noted above, there seems to be only one flaw in the book. This is the author's tendency to mis-state or omit various assumptions; the presence of a unit, completeness under a norm, continuity of a representation, and so on must often be checked by the reader. A reader needing theorems for applications will need to make sure that he is getting a correct theorem. No doubt these lacunae can be corrected in a second edition, and in the meantime, Professor Naĭmark has produced a unique, monumental, and highly valuable work. *E. Hewitt* (Seattle, Wash.).

Stanojević, Časlav V. Note on regular elements in an extension of Banach algebra without identity. Bull. Soc. Math. Phys. Serbie **8** (1956), 183–190. (Serbo-Croatian. English summary)

La plus grande partie de ce travail concerne l'immersion d'une algèbre normée B sur le corps K des complexes et

sans élément unité dans une K-algèbre A avec unité et les rapports entre l'inversibilité dans A et la quasi-inversibilité dans B. Tout cela est bien connu (et classique). En particulier, l'adjonction de l'unité décrite par l'auteur se trouve explicitement définie, dans le cas algébrique abstrait, dans l'exerc. 1, § 8, Chap. I, de l'"Algèbre" de N. Bourbaki, [Actualités Sci. Ind., no. 934, Hermann, Paris, 1942; MR 6, 113] et au § 7, p. 147 des "Anneaux normés" (en russe) de M. A. Naïmark [voir l'analyse ci-dessus] et, dans le cas des anneaux normés, au § 9, p. 163 du même livre.

La seule partie peut-être inédite (mais assez immédiate) du travail est la transcription de certaines résultats connus sur les résolventes $R(\lambda, a)=(a-\lambda\cdot e)^{-1}$ des $a\in A$, où e est l'unité de A [tels: (1) la résolvente de a est définie et holomorphe en λ si $|\lambda|>\|a^n\|^{1/n}$; (2) elle est une fonction entière de λ si $\|a^n\|^{1/n}\to 0$], en termes de $b=a-e$ et de la quasi-inversibilité dans B quand $a\in e+B$. *M. Krasner.*

Hongo, Eishi. On quasi-unitary algebras with semi-finite left rings. Bull. Kyushu Inst. Tech. (Math. Nat. Sci.) no. 3 (1957), 1–10.

A complex algebra R is called "quasi-unitary" [cf. Dixmier, Comment. Math. Helv. **26** (1952), 275–322; MR **14**, 660] if it has an involutive anti-automorphism $x\to x^e$, an automorphism $x\to x^j$ and an inner product (x, y) such that: (i) $(x^e, x^e)=(x, x)$; (ii) $(x, x^j)\geqq 0$; (iii) $(xy, z)=(y, x^{ej}z)$; (iv) the left multiplications $U_x: y\to xy$ are continuous; and (v) elements of the form $xy+(xy)^j$ are dense in R. If $x^j=x$, then R is called "unitary". If $\mathfrak{H}_R$ is the Hilbert space completion of R, then each of the operators U_x has a unique extension to $\mathfrak{H}_R$. The weakly closed self-adjoint algebra of bounded operators on $\mathfrak{H}_R$ generated by the left multiplications is denoted by R^e. The main result in the paper is the theorem: If R is a quasi-unitary algebra with semi-finite left ring R^e, then there exists in $\mathfrak{H}_R$ a dense quasi-unitary algebra R' which can be renormed (not necessarily equivalently) in such a way that it becomes a unitary algebra with the same involution. A converse to this theorem is also obtained. These results generalize results obtained by Ogasawara [J. Sci. Hiroshima Univ. Ser. A. **19** (1955), 79–85; MR **17**, 1227]. *C. E. Rickart* (New Haven, Conn.).

Umegaki, Hisaharu. Conditional expectation in an operator algebra. II. Tôhoku Math. J. (2) **8** (1956), 86–100.

The notion of "conditional expectation" in probability theory has a non-commutative generalization, in the following sense: given a semifinite von Neumann algebra A with regular gage μ, and a von Neumann subalgebra A_1 which is generated by projections on which μ is finite, then Dixmier has shown that there is a unique linear positivity-preserving idempotent operator on A whose fixed set is A_1, and which satisfies certain axioms which can be regarded as a generalization of Moy's axioms for conditional expectation in the commutative case. In the case $\mu(I)<\infty$, Umegaki [same J. (2) **6** (1954), 177–181; MR **16**, 936] had in a previous paper given necessary and sufficient conditions for a linear operator on A that it be the conditional expectation with respect to some von Neumann subalgebra. Here he gives a characterization in the semifinite case, with analogous characterizations of the corresponding operators on $L_1(A)$ and $L_2(A)$. The notion of "martingale" is also generalized to what the author calls an increasing (decreasing) M-net: an upward (downward) directed set of operators X_α in A (or $L_1(A)$, or $L_2(A)$) such that if A_α is the least subalgebra of A with which all X_γ, $\gamma<\alpha$, are affiliated, then $\alpha<\beta\Rightarrow X_\alpha$ is the conditional expectation of X_β with respect to A_α. The M-net is called simple if there is a fixed X such that X_α is the conditional expectation of X with respect to A_α. When A is finite, various characterizations and properties of simple nets are given; for example, if X is associated with the increasing simple net X_α, and $X\in L_1(A)$, then $X_\alpha\to X$ in $L_1(A)$. These ideas are related to some work of I. E. Segal; the above theorem, for example, was proven independently in "Inductive Limits of Probability Spaces" [mimeographed but unpublished]. Finally, the discussion of decreasing simple M-nets is applied to give an alternate treatment of von Neumann's Th. 6 in Ann. of Math. (2) **41** (1940), 94–161 [MR **1**, 146]. *J. Feldman.*

Nakamura, Masahiro; and Umegaki, Hisaharu. On a proposition of von Neumann. Kōdai Math. Sem. Rep. **8** (1956), 142–144.

Let M be a semi-finite W^*-algebra with a regular gage μ and denote by M_μ the μ-integrable elements of M. If A is any W^*-subalgebra of M, then there exists a unique non-negative linear mapping $x\to x^e$ of M into A such that $x^e=I^e x$, for $x\in A$, $x^{*e}x^e\leqq(x^*x)^e$, $x^{ee}=x^e$, and $\mu(x^e y)=\mu(xy^e)$ for all $x\in M$ and $y\in M_\mu$ [Dixmier, Bull. Soc. Math. France **81** (1953), 9–39; MR **15**, 539]. The operation $x\to x^e$ is called the "conditional expectation (of x) conditioned by A" [see the paper reviewed above]. If M acts on a separable Hilbert space and A is an abelian W^*-subalgebra of M generated by a sequence $p, q, \cdots$ of projections, then von Neumann has defined an operation $x\to x^{|p|q|\cdots}$ which maps M_μ into A and asserted without proof that this mapping depends only on x and A but not on the choice of $p, q, \cdots$ [in his paper cited in review above]. The present paper contains a proof of this assertion. The authors also show in this situation that the abelian W^*-algebra A is maximally abelian if and only if the conditional expectation conditioned by A coincides with von Neumann's operation defined by A.

C. E. Rickart (New Haven, Conn.).

Kostučenko, A. G. On the behaviour of the eigenfunctions of selfadjoint operators. Dokl. Akad. Nauk SSSR (N.S.) **114** (1957), 249–251. (Russian)

Motivated by the problem of studying whether or not the solutions of $-y''+q(x)y=\lambda y$, $-\infty<x<\infty$, $q(x)>c>-\infty$, are bounded, the author investigates the eigenfunctions of the general Hermitian operator A over $L_2(R_n)$. The basic assumption is that the resolvent, R_λ of A is, for at least one value λ_0 of λ, an integral operator: $R_{\lambda_0}f=\int_{-\infty}^{\infty}K(x, y)f(y)dy$, where $\int_{-\infty}^{\infty}|K^2(x, y)|dy<C$, Cx-free. Under these circumstances, let $\{E_\lambda\}$ be the resolution of the identity of A, let $g^{(\alpha)}(x)$ be cyclic elements, $L_2^{(\alpha)}$ the associated cyclic subspaces on which A is simple and of which $L_2(R_n)$ is the direct sum. Let $\sigma_\alpha(-\infty, \lambda)\equiv\sigma_\alpha(\lambda)=(E_\lambda g^{(\alpha)}, g^{(\alpha)})$. Then the functions $d(E_\lambda g^{(\alpha)}(x))/d\sigma_\alpha(\lambda)$ are bounded for almost every λ (relative to the measure $\sigma_\alpha(\lambda)$). Since these functions are "ordinary" functions as opposed to "generalized" functions (distributions) under the hypotheses mentioned, and since they are also eigenfunctions of A, their boundedness is both meaningful and relevant. The basic tool is Gel'fand's lemma on the differentiation of functionals (over L_1) of strongly bounded variation [Mat. Sb. N.S. **4(46)** (1938), 235–284]. Applications to the Schrödinger equation are indicated.

B. R. Gelbaum (Minneapolis, Minn.).

Poliatsky, V. T. On the reduction of quasi-unitary operators to a triangular form. Dokl. Akad. Nauk SSSR (N.S.) **113** (1957), 756–759. (Russian)

By reference to the work of M. S. Livšic [Mat. Sb. N.S. **34**(76) (1954), 145–199; MR **16**, 48] and V. P. Potapov [Trudy Moskov. Mat. Obšč. **4** (1955), 125–236; MR **17**, 958], the author shows how to triangularize a semi-unitary operator T (i.e. a T such that $I-TT^*$ and $I-T^*T$ have ranges of finite, equal dimension) by means of a unitary transformation. The basic tool is the characteristic matrix function of Livšic and its special (Blaschke product type of) representation due to Potapov. Given the ingredients of some Potapov representation, the author constructs an operator with the required characteristic matrix function, and the form of the operator is appropriately "triangular." Conversely, given the semi-unitary operator, the author passes through the Potapov representation of its characteristic matrix function to some triangular form, as in the preceding sentence. The unitary transformation is then obvious. Applications to special situations are indicated. *B. R. Gelbaum.*

Harazov, D. F. On some properties of linear operators ensuring the correctness of the Hilbert-Schmidt theorem. Uspehi Mat. Nauk (N.S.) **12** (1957), no. 4(76), 201–207. (Russian)

Extending results of H. Wielandt [Math. Nachr. **4** (1951), 308–314; MR **12**, 717], it is proved that: if X is a complex space with semidefinite scalar product, not necessarily complete, H a positive operator in X, and A a symmetrisable operator such that $(HAx, y)=(x, HAy)$ for all x, y, and such that (1) the spectrum of A consists, at most, of points of finite multiplicity with no finite limit point, (2) $HA\neq 0$, and (3) if λ is a proper value, $x-\lambda Ax=0$, then $(Hx, x)\neq 0$ {Reviewer's note: the terms spectrum and proper value are used by the author in the sense defined by this equation}; then the Hilbert-Schmidt theory applies to A in the following way: (A) there is at least one proper value; (B) all proper values are real, $(Hx, y)=0$ for x, y proper elements for different proper values; (C) if the proper elements are ordered in increasing order of modulus, then the maximum of $|(HAx, x)|$ for x such that Hx is orthogonal to the first $(n-1)$ characteristic elements is $1/|\lambda_n|$; (D) $\sum \lambda_k^{-1}(f, Hx_k)Hx_k$ converges weakly to HAf, and converges strongly if H is bounded; and (E) the solution of $x-\lambda Ax=y$ is given by a formula analogous to Schmidt's:

$$Hx=\lambda\sum (\lambda_k-\lambda)^{-1}\ (y,Hx_k)\ Hx_k+Hy.$$

Applications to the theory of symmetrisable operators and to the theory of equations $M(y)=\lambda N(y)-f(x)$, where M and N are differential operators, are given.

J. L. B. Cooper (Cardiff).

Sz.-Nagy, Béla. Sur les contractions de l'espace de Hilbert. II. Acta Sci. Math. Szeged **18** (1957), 1–14.

Let H be a Hilbert space and let T be a contraction, i.e., a bounded linear operator on H with $\|T\|\leq 1$. The author has previously shown [same Acta **15** (1953), 87–92; MR **15**, 326] that there exists a larger Hilbert space K, $H\subset K$, and a unitary operator U on K such that $T^{(n)}=\mathrm{pr}U^{(n)}$, $n=0, \pm 1, \pm 2, \cdots$, where K is determined by the elements $\{U^n h\}$, $h\in H$ and U is determined in a unique manner. In the present paper a new proof of a theorem of Schreiber [Duke Math. J. **23** (1956), 579–594; MR **18**, 748] is given. Let T be a proper contraction (i.e. $\|T\|<1$). Then the transformation U is unitarily equivalent to the d-fold orthogonal sum of the unitary operator $V[u(\varphi)]=e^{i\varphi}u(\varphi)$, where $U\in L^2(0, 2\pi)$ and where d is the dimension of H. The method of the author is more general than that of Schreiber in that it is not restricted to the case $d\leq\aleph_0$. A generalization is also given in which a similar representation theorem is given for a one-parameter semigroup, $T(s)$, of proper contractions. In this case the representation yields a d-fold orthogonal sum of the one-parameter group $V(s): L^2(-\infty, \infty)\to L^2(-\infty, \infty)$ where $V(s)[u(\varphi)]=e^{i\varphi s}U(\varphi)$. *R. E. Fullerton* (College Park, Md.).

See also: Approximations, Orthogonal Functions: Kazmin. **Ordinary Differential Equations:** Višik and Lyusternik. **Partial Differential Equations:** Phillips. **Integral and Integrodifferential Equations:** Sakhnovich; Yoshimatsu. **Topological Vector Spaces:** Shimogaki. **Complex Manifolds:** Bremermann. **Numerical Methods:** Mihlin; Kantorovitch; Polak.

TOPOLOGY

General Topology

★ **Kuratowski, Casimir. Topologie. Vol. I. 4ème éd.** Monografie Matematyczne, Tom 20. Państwowe Wydawnictwo Naukowe, Warsaw, 1958. xiii+494 pp.

This is essentially a reprint of the 3rd edition [MR **14**, 1000; for 2nd edition see MR **10**, 389], with an appendix in which the author gives a brief account of some of the topics not contained in earlier editions. These are, in particular, the concepts of completely regular space, bicompact space, and the Cartesian product of Tychonov. The appendix also contains two short notes, one by A. Mostowski on some applications of topology to mathematical logic, the other by R. Sikorski on the applications of topology to functional analysis.

Iséki, Kiyoshi. A remark on countably compact normal space. Proc. Japan Acad. **33** (1957), 131–133.

Nine conditions equivalent to countable compactness for a normal space are given, all dealing with the existence of finite subcoverings for coverings of various types. *E. Hewitt* (Seattle, Wash.).

Iséki, Kiyoshi. On AU-property and countably compactness. Proc. Japan Acad. **33** (1957), 357.

The author shows that a space constructed by Ramanathan [Proc. Indian Acad. Sci. Sect. A. **26** (1947), 31–42; MR **9**, 98] is a semi-regular space that is not countably compact but still has the AU-property (every countable open covering admits a finite subfamily whose closures cover the space). *E. Hewitt* (Seattle, Wash.).

Iséki, Kiyoshi. AU-covering theorem and compactness. Proc. Japan Acad. **33** (1957), 363–367.

This paper contains miscellaneous facts about feeble compactness [see Mardešić and Papić, Hrvatsko Prirod. Društvo. Glasnik Mat.-Fiz. Astr. Ser. II. **10** (1955), 225–232; MR **18**, 224], the AU-property [see preceding review], and absolute closure. Sample theorem: for a regular space S, the following conditions are equivalent: 1) S is feebly compact; 2) every locally finite open covering contains an AU-covering; 3) every countably infinite open covering contains an AU-covering; 4) every locally finite open covering contains a finite subcovering. There is also an interesting characterization of absolute closure for Hausdorff spaces. *E. Hewitt* (Seattle, Wash.).

Iséki, Kiyoshi. Pseudo-compactness and μ-convergence. Proc. Japan. Acad. **33** (1957), 368–371.

Let S be a set. A sequence $\{f_n\}_{n=1}^{\infty}$ of real-valued functions on S is said to converge quasi-uniformly on S to a function f if it converges pointwise to f, and if for every $\varepsilon>0$ and every positive integer N, there are positive integers $n_1, \cdots, n_k \geq N$, such that, for every $x \in S$, $\min\{|f_{n_i}(x)-f(x)| : i=1, \cdots, k\}<\varepsilon$. Following the method of G. Sirvint [Studia Math. **11** (1950), 71–94; MR **14**, 183], the author says that $\{f_n\}_{n=1}^{\infty}$ μ-converges to 0 if for every $\varepsilon>0$ there are non-negative numbers $\lambda_1, \cdots, \lambda_k$ such that $\sum_i \lambda_i=1$ and $|\sum_i \lambda_i f_i(x)|<\varepsilon$ for all $x \in S$. Theorem: For a completely regular space S, the following conditions are equivalent: 1) S is pseudo-compact; 2) if a sequence of continuous real-valued functions on S converges to 0, then it converges quasi-uniformly to 0; 3) if a decreasing sequence of functions converges to 0, then it μ-converges to 0. *E. Hewitt* (Seattle, Wash.).

Hayashi, Yoshiaki. On Dowker's problem. Proc. Japan Acad. **33** (1957), 351–354.

The problem [Canad. J. Math. **3** (1951), 219–224; MR **13**, 264] is to find a normal Hausdorff space which is not countably paracompact. A would-be example is given here, but unfortunately it is not normal. {On p. 352, line 12, the assertion $\alpha<\Omega$ is erroneous; the possibility $\alpha_n=\Omega$ has been overlooked.} *A. H. Stone.*

Katětov, Miroslav. On the dimension of non-separable spaces. II. Czechoslovak Math. J. **6(81)** (1956), 485–516. (Russian. English summary)

This continuation of an earlier paper [same J. **2(77)** (1952), 333–368; MR **15**, 815] is divided into three sections. Section 1 contains several theorems asserting that almost all maps from a given space to a separable Banach space have nice properties of various types. Let R be a separable Banach space and let P be a normal space; $C'(P, R)$ denotes the set of all totally bounded functions from P to R. Then, if S is closed in P and $\dim S \leqq \text{lin dim } R$, for almost all $f \in C'(P, R)$ in the sense of category, $\dim \overline{f(S)}=\dim f(S)=\dim S$. Also, if S is a closed G_δ in P and $f \in C'(P, R)$, for almost all $F \in C'(P, R)$ which agree with f on S we have $\dim F(P-S) \leqq \dim (P-S)$. Other results of the same general type are proved. Section 2 contains various results concerning the existence of arbitrarily fine coverings of a metric space P whose intersection properties with each element of a given sequence $\{A_i\}$ of closed subsets of P are nice. This section has considerable contact with the work of Morita [Math. Ann. **128** (1954), 350–362; MR **16**, 501]. The last section concerns the existence of extensions of maps with the extended image differing from the original image by a set of small dimension. For example, it is proved that if P, S, R are as above, and if K is a convex subset of R, every $f:S \to K$ has an extension $F:P \to K$ with $\dim F(P-S) \leqq \min[\dim S+1, \dim f(S)+1, \dim P]$. *E. E. Floyd.*

Strother, W. L.; and Ward, L. E., Jr. Retracts from neighborhood retracts. Duke Math. J. **25** (1957), 11–14.

A compact Hausdorff space X is called a CAR* [CANR*] if it is a retract [neighborhood retract] of a Tychonoff cube; for metric X, this is equivalent to being an AR [ANR]. Let $S(X)$ denote the space of non-empty, closed subsets of X, in the usual topology. Theorem 1: If X is a metrizable, connected CANR*, then $S(X)$ is a CAK*. Theorem 2: If X is a connected CANR*, then $S(X)$ has the fixed point property. {Reviewer's comments: (1) Theorem 1 is a special case of Theorem II of Wojdysławski [Fund. Math. **32** (1939), 184–192]. (2) The introductory remark that L. Foulis [Proc. Amer. Math. Soc. **8** (1957), 365–366; MR **18**, 750] answered a question of Wojdysławski [op. cit.] is erroneous; Wojdysławski asks no questions at all in that paper, and asks no question answered by L. Foulis anywhere.} *E. Michael.*

Polak, A. I. On the mechanism of uniform approximations in the domain of continuous maps of compacts. Uspehi Mat. Nauk (N.S.) **11** (1956), no. 4(70), 149–154. (Russian)

Suppose X and Y are compact metric spaces, and that $\{f_n\}$ is a sequence of continuous maps of X into Y which converges uniformly to $f:X \to Y$. Various theorems are proved which deal with the closeness of $f_n^{-1}(y)$ and $f^{-1}(y)$. If f is open, it is shown that given $\varepsilon>0$ there is an $N(\varepsilon)$ with $f_n^{-1}(y) \subset N_\varepsilon(f^{-1}(y))$ for all $y \in Y$ and $n \geqq N$. The sequence $\{f_n\}$ is defined to be uniformly open if and only if, given $x \in X$ and $\varepsilon>0$, there exists $\delta(x, \varepsilon)$ with $f_n(N_\varepsilon(x)) \supset N_\delta(f_n(x))$ for all n. If now f_n and f are open, then a necessary and sufficient condition that the Hausdorff distance $\rho(f_n^{-1}(y), f^{-1}(y))$ approach 0 uniformly with n is that the sequence $\{f_n\}$ be uniformly open. Other results of a similar type are given. *E. E. Floyd.*

de Groot, J. On a metric that characterizes dimension. Canad. J. Math. **9** (1957), 511–514.

A topological space M is a separable metric space of dimension $\leqq n$ if and only if there is a totally bounded metric ρ on M such that for every $n+3$ points $x, y_1, y_2, \cdots, y_{n+2}$ in M there are three indices i, j, k such that $\rho(y_i, y_j) \leqq \rho(x, y_k)$ $(i \neq j)$. For compact metric spaces M, the restriction of total boundedness can obviously be dropped. This theorem is a simplification of, and is proved from, a theorem of J. Nagata (which, however, is valid for all metric spaces) [Proc. Japan Acad. **32** (1956), 237–240; MR **19**, 156]. *E. Hewitt* (Seattle, Wash.).

Keldyš, Lyudmila. Monotone mapping of a cube onto a cube of higher dimension. Mat. Sb. N.S. **41(83)** (1957), 129–158. (Russian)

Detailed account of a result announced earlier [Dokl. Akad. Nauk SSSR (N.S.) **103** (1955), 957–960; MR **17**, 992]. *E. E. Floyd* (Charlottesville, Va.).

Wunderlich, W. Irregular curves and functional equations. Ganita **5** (1954), 215–230 (1955).

The author observes that there can be abstracted from descriptions of several continuous nowhere-differentiable functions and from descriptions of certain other pathological curves in euclidean spaces a common "principle of generation" involving an integer parameter $r \geqq 2$. He further notes that the representation of real numbers in terms of this r as base is advantageous in analytic formulations of such descriptions. *T. A. Botts.*

See also: Foundations, Theory of Sets, Logic: Sikorski. **Measure, Integration:** Cesari and Fullerton.

Algebraic Topology

Palermo, F. P. The cohomology ring of product complexes. Trans. Amer. Math. Soc. **86** (1957), 174–196.

It is known that the integral cohomology rings $H(X)$,

$H(Y)$ of two spaces X, Y are insufficient to determine the multiplicative structure of the cohomology ring $H(X\times Y)$. The author shows that if one has additional information, namely the cohomology spectra of X and Y, then this is sufficient to determine not only the cohomology ring $H(X\times Y)$ but the whole cohomology spectrum of $X\times Y$. The cohomology spectrum of a cochain complex K is defined to be the cohomology ring $H(K)$, together with the cohomology rings $H(K, n)$ with coefficients modulo n, the coefficient homomorphisms $h_{n,0}:H(K)\to H(K, n)$ and $h_{n,m}:H(K, m)\to H(K, n)$, and the Bockstein homomorphisms $\Delta_n:H(K, n)\to H(K)$ of degree $+1$.

For K, L free and finitely generated, an explicit construction is given for the spectrum of $K\otimes L$ from those of K, L. The result is extended to the case when K, L are infinite torsion free by taking direct limits, and is applied to Čech cohomology to obtain the theorem above.

An interesting example shows that the Bockstein and coefficient homomorphisms are necessary: the lens spaces $L(5, 1)$ and $L(5, 2)$ have isomorphic cohomology rings for all coefficients, but $L(5, 1)\times L(5, 1)$ and $L(5, 2)\times L(5, 2)$ do not.

E. C. Zeeman (Cambridge, England).

Bokšteĭn, M. F. Homological invariants of topological spaces. II. Trudy Moskov. Mat. Obšč. **6** (1957), 3–133. (Russian)

Part I of this work has appeared earlier [same Trudy **5** (1956), 3–80; MR **18**, 813]. For notation and terminology, see the review cited. Chapters III and IV, comprising Part II, deal with homological invariants of product spaces. In Ch. III, the ∇-groups (∇-rings) of the Cartesian product $A\times B$ of two locally compact Hausdorff spaces A and B are computed in terms of the complete modular spectra of ∇-rings (∇-groups) of the spaces A and B. The complete modular spectrum of ∇-groups of a locally compact Hausdorff space A is the set of all ∇-groups mod m ($m=0, 1, 2, \cdots$) of all possible dimensions, together with the homomorphisms $\pi_m^{m'}$ and $\check{\omega}_{m'}^{m}$, defined above and a set of homomorphisms χ_0^m carrying the q-dimensional cohomology group mod m, $\nabla_m^q(A)$, into the $(q+1)$-dimensional integer cohomology group $\nabla_0^{q+1}(A)$. The complete modular spectrum of $\bar{\nabla}$-rings is defined similarly. The technique is to construct an isomorph of $\nabla_m^q(A\times B)$ from a knowledge of the groups and homomorphisms for A and B separately. The computations are extraordinarily long.

Ch. IV deals with dimension (cohomology dimension) for the Cartesian product of two spaces as above. The main theorem gives the value of $\mathrm{Dim}_G(A\times B)$ in terms of $\mathrm{Dim}_G A$ and $\mathrm{Dim}_G B$ for the four classes of groups G dealt with in Ch. II. For example, writing R for the additive rationals, C_p for the cyclic group of order p, and Q_p for the p^∞ group (p a prime), we have $\mathrm{Dim}_R(A\times B)=\mathrm{Dim}_R A+\mathrm{Dim}_R B$, and $\mathrm{Dim}_{Q_p}(A\times B)=\max(\mathrm{Dim}_{Q_p} A+\mathrm{Dim}_{Q_p} B, \mathrm{Dim}_{C_p} A+\mathrm{Dim}_{C_p} B-1)$. Once again, the computations are somewhat forbidding. *E. Hewitt* (Seattle, Wash.).

Ando, Hideo. A note on Eilenberg-MacLane invariant. Tôhoku Math. J. (2) **9** (1957), 96–104.

The author generalizes the definition of the Eilenberg-MacLane invariant [Ann. of Math. (2) **51** (1950), 514–533; MR **11**, 735], using the notion of group-system [Blakers, ibid. **49** (1948), 428–461; MR **9**, 457], and then generalizes the result of Eilenberg-MacLane on this invariant which says that it determines the homology groups for an appropriate range of dimensions. *N. Stein.*

★ **Schwartz, Marie-Hélène. Espacios fibrados. [Fibre spaces.]** Universidad Nacional de Colombia. Departamento de Matemáticas y Estadística. Bogota, Colombia, 1956. 58 pp. (mimeographed) $1.00.

This is a set of mimeographed lecture notes from a course given by the author. Part I, entitled "General Theory of Fibre Spaces" (pp. 1–35), gives the basic definitions and some of the elementary theorems about fibre spaces. Part II (pp. 36–56) is entitled "Topological study of sections of a fibre space." It is concerned with the theory of obstructions to a cross-section of a fibre space. The main theorem proved here is that the first obstruction to a cross-section is the negative of the transgression of the fundamental class of the fibre. Stiefel-Whitney classes and Chern classes are mentioned briefly. *W. S. Massey* (Providence, R.I.).

Adachi, Masahisa. Sur les groupes de cobordisme Ω^k. Proc. Japan Acad. **33** (1957), 143–144.

The cobordism groups Ω^k, $k\geqq 0$, were defined by R. Thom [Comment. Math. Helv. **28** (1954), 17–86; MR **15**, 890]. Thom himself computed the structure of the groups Ω^k for $k<8$. In the present paper, the author computes their structure for $k=8$, 9, and 10. His results are as follows: $\Omega^8=Z+Z$, $\Omega^9=Z_2+Z_2$, and $\Omega^{10}=Z_2$. In making these computations, he uses the methods introduced by Thom [loc. cit.].

The determination of Ω^8 leads to the following corollary: If all the Pontrjagin numbers of a compact, differentiable, oriented 8-dimensional manifold are zero, then the manifold is a bounding manifold. *W. S. Massey.*

James, I. M. Multiplication on spheres. II. Trans. Amer. Math. Soc. **84** (1957), 545–558.

[For Part I see Proc. Amer. Math. Soc. **8** (1957), 192–196; MR **18**, 752.] A multiplication: $A\times A\to A$ ($(x, y)\to x\cdot y$) on the space A is called homotopy associative if the two maps, g, h: $A\times A\times A\to A$ defined by $g(x, y, z)=(x\cdot y)\cdot z$, $h(x, y, z)=x\cdot(y\cdot z)$ are homotopic. The main result of this paper is that, of the spheres, only S^1 and S^3 admit a homotopy associative multiplication. The author also describes all the possible multiplications on these spheres. This is done by means of the following general result: If S^n admits a multiplication, then the homotopy classes of multiplications are in one to one correspondence with the elements of $\pi_{2n}(S^n)$. Thus S^1 has essentially only one multiplication and S^3 has 12. (Of these it turns out that 8 are "associative" and 4 are not.). The principal tool of the paper is the "separation element", $d(u, v)$, of two maps of a product of spheres, M, into a space A. If Σ is the complement of the top cell in M, the separation element is defined whenever u and v agree on Σ, and then $d(u, v)$ is a class in $\pi_m(A)$; $m=\dim M$ [cf. the paper reviewed below]. The author first shows that, if A is a sphere which admits a multiplication, then two maps $u, v:M\to A$ which agree on Σ are homotopic if and only if their separation element vanishes. Now, if S^n admits a multiplication, the two maps g and h defined earlier determine a separation element $d\in\pi_{3n}(S^n)$, and the multiplication is homotopy associative if and only if $d=0$.

The main result is derived from the following general formula which is valid for spheres which admit a multiplication $f:S\times S\to S$. Letting $c(f)$ denote the map of S^{2n+1} into S^{n+1} obtained by the Hopf construction from f, the author shows that $Ed(f)=Pc(f) \mod F\pi_{3n+1}(S^{2n+1})$, where F is induced by $c(f)$, E is the suspension, and P denotes Whitehead product with a generator of $\pi_{n+1}(S^{n+1})$. *R. Bott* (Ann Arbor, Mich.).

James, I. M. On spaces with a multiplication. Pacific J. Math. **7** (1957), 1083–1100.

In this paper the author continues his study of the suspension $\hat{A}$ of a space A. First, if Z is any space with a multiplication, he associates an element $\delta(h) \in \pi_{p+q}(Z)$ with a map h: $S^p \times S^q \to Z$, and defines a product $\pi_p(Z) \otimes \pi_q(Z) \to \pi_{p+q}(Z)$. Taking Z to be A_∞, the reduced product space of A, he gives a new characterization of the Hopf construction. Finally, he studies the relations among the Whitehead product in A and the product mentioned above in A_∞ and in the space of loops on $\hat{A}$. Using these relations, he gives a new proof of the first commutation law for triad Whitehead products. *N. Stein.*

Read, R. C. Maximal circuits in critical graphs. J. London Math. Soc. **32** (1957), 456–462.

With a critical k-chromatic graph of order n, where $k \geqq 3$ and $n \geqq k$, the length of the longest circuit(s) contained in the graph may be associated. For given $k \geqq 3$ and $n \geqq k$, $L_k(n)$ is defined as the minimum of the lengths of the longest circuits of all critical k-chromatic graphs of order n. It is known that $L_k(n)$ tends to infinity with n for each $k \geqq 3$ [J. B. Kelly and L. M. Kelly, Amer. J. Math. **76** (1954), 786–792; MR **16**, 387]. The present paper makes a new contribution to the investigation of the order of magnitude of $L_k(n)$ by showing that, for each $k \geqq 4$,

$$\liminf L_k(n)/\{\log n \times \log\log n \times \log\log\log n \times \cdots \times \log_{(k-4)} n \times [\log_{(k-3)} n]^2\} < \left[\frac{2}{\log 4}\right]^2$$

where $\log_{(m)} n$ denotes the m-times iterated logarithm.

In order to construct critical k-chromatic graphs with small maximal circuits, the author introduces a new technique for colouring the nodes. They are first coloured arbitrarily using a stock of primary colours, and then, where necessary, some nodes are re-coloured using a stock of secondary colours in such a way that two nodes which are joined by an edge do not have the same colour.

G. A. Dirac (Hamburg).

Kotzig, Anton. Aus der Theorie der endlichen regulären Graphen dritten und vierten Grades. Časopis Pěst. Mat. **82** (1957), 76–92. (Slovak. Russian and German summaries)

Ist G ein (nicht gerichteter) endlicher regulärer Graph dritten Grades, so sei G^* der Graph, dessen Knotenpunkte den Kanten von G eineindeutig entsprechen, wobei zwei solche Knotenpunkte durch eine Kante in G^* dann und nur dann verbunden sind, falls die entsprechende Kanten von G benachbart sind. G^* ist ein regulärer Graph vierten Grades. Die Anzahl seiner Knotenpunkte sei gerade. Dann besitzt er einen Faktor ersten Grades, und falls auch G einen Faktor ersten Grades enthält, besitzt G^* einen Faktor zweiten Grades, der in zwei Faktoren ersten Grades zerfällt. Ist G in drei Faktoren ersten Grades zerlegbar, so ist G^* in vier Faktoren ersten Grades zerlegbar und umgekehrt. Existiert in G eine Hamiltonsche Linie, dann zerfällt G^* in zwei Hamiltonsche Linien und umgekehrt. *M. Fiedler* (Praha).

See also: Combinatorial Analysis: Riordan. **Lie Groups and Algebras:** Kojima.

GEOMETRY

Geometries, Euclidean and Other

de Goñi Peralta, Pedro. A triangle whose inscribed circle passes through its barycenter. Euclides, Madrid **17** (1957), 78–79. (Spanish)

Zappa, Guido. Sui gruppi di collineazioni dei piani di Hughes. Boll. Un. Mat. Ital. (3) **12** (1957), 507–516.

The author studies the collineation groups of the non-Desarguesian projective planes recently introduced by the reviewer [Canad. J. Math. **9** (1957), 378–388; MR **19**, 444], which he calls Hughes planes. A plane π of this class has order p^{2n}, p an odd prime, n a positive integer, contains a Desarguesian subplane π_0 of order p^n, and can never be coordinatized by a Veblen-Wedderburn system. It is shown that π possesses the group of projectivities of π_0 as a collineation group, as well as the collineation group induced by the automorphisms of the near-field which underlies the construction of π. In the case of the Hughes plane of order 9, it is shown that the direct product of these two groups is indeed the complete collineation group of the plane. *D. R. Hughes* (Columbus, Ohio).

Hughes, D. R. A note on some partially transitive projective planes. Proc. Amer. Math. Soc. **8** (1957), 978–981.

Using the notation of Amer. J. Math. **78** (1956), 650–674 [MR **18**, 921], the author raises the question of the existence of a projective plane of "type $(4, m)$" with $m=3$, $n \neq 4$. Using results of Baer and Paige, he shows that such a plane does not exist. Also, he proves that for every $n=p^{2i} \neq 4$, p a prime, there is a plane of type $(4, n)$.

S. Chowla (Princeton, N.J.).

Havel, Václav. Über die lokalen Spezialisierungen des Satzes vom vollständigen Viereck und des kleinen Desarguesschen Satzes. Czechoslovak Math. J. **7**(82) (1957), 295–307. (Russian summary)

Consider a projective plane without quadrangles with collinear diagonal points. The author analyzes some relations between certain specializations of the following propositions: (Q) if A, B are two diagonal points of two plane quadrangles and C is a point collinear with A, B belonging to a side of each of the quadrangles, the intersection D of the other sides of these quadrangles is a point collinear with A, B; (d) the theorem of Desargues on perspective triangles. There are also given conditions for the validity of the equation $(-1)u=-u$ for every element u of the ground field of the projective plane.

P. Abellanas (Madrid).

★ **Hobson, E. W. A treatise on plane and advanced trigonometry. 7th ed.** Dover Publications, Inc., New York, 1957. xv+383 pp. $1.95.

A reprinting, unchanged except for title, of the author's book first published by Cambridge in 1891 under the title A treatise on plane trigonometry.

Terracini, Alessandro. Un'osservazione su un passo di un lavoro giovanile di Corrado Segre. Boll. Un. Mat. Ital. (3) **12** (1957), 673–677.

See also: Manifolds, Connections: Gołąb. **Algebraic Geometry:** Thalberg. **Relativity:** Schmutzer.

Convex Domains, Integral Geometry

Stein, Sherman K. A continuous mapping defined by a convex curve. Math. Z. **68** (1957), 282–283.

Let K be a closed, bounded, convex, plane region, whose boundary B is differentiable and does not contain any straight line segments. If Q is a point of the plane for which distinct tangents to B, QA_1, QA_2, exist with points of contact A_1, A_2, then $g(Q)$ is defined so that Q, A_1, A_2, $g(Q)$ are the four vertices of the parallelogram with sides QA_1, QA_2. If $Q \in B$, then $g(Q)$ is defined to be Q. The author shows: (1) that a point P exists so that $g^{-1}(P)$ contains an arbitrary point of the plane; (2) that $g^{-1}(P)$ contains at least two points if $P \in K$; and (3) that at least one point P exists in K so that $g^{-1}(P)$ contains at least four points. *D. Derry* (Vancouver, B.C.).

Blundon, W. J. Multiple covering of the plane by circles. Mathematika **4** (1957), 7–16.

Let $\vartheta^{(r)}$ denote the density of the thinnest r-fold lattice-covering of the plane by congruent circles. It is known that $\vartheta^{(1)}=\vartheta=2\pi\cdot 3^{-3/2}$. The author obtains the values of $\vartheta^{(2)}$, $\vartheta^{(3)}$ and $\vartheta^{(4)}$ and determines the lattices for thinnest covering in these cases. He states that no regular pattern seems to emerge. As general results he has the simple facts: $\vartheta^{(r)} \leqq r\vartheta$ {he could have said $\vartheta^{(rs)} \leqq r\vartheta^{(s)}$} and $\vartheta^{(r)}/r \to 1$. *E. G. Straus* (Los Angeles, Calif.).

Rogers, C. A. A note on coverings. Mathematika **4** (1957), 1–6.

Let ϑ^* denote the density of the most economical covering of E^n by the translates of a bounded convex body K. The author proves $\vartheta^* \leqq \theta_n^*$, where

$$\theta_n^* = \min_{0<\eta<1/n} (1+\eta)^n\{1-n\log\eta\} < n\log n + n\log\log n + 5n$$

for $n \geqq 3$. This is a considerable improvement of previous estimates which were of exponential order in n.

The method of proof is based on the fact that a random covering of a rectangular box of side length $R>\operatorname{diam} K$ by N translates of K leaves a complement of density $(1-R^{-n})^N$. *E. G. Straus* (Los Angeles, Calif.).

Rogers, C. A. The compound of convex bodies. J. London Math. Soc. **32** (1957), 311–318.

Let $1 \leqq p \leqq n-1$ and let $X^{(\pi)}=(x_{\pi 1}, \cdots, x_{\pi n})$ $(\pi=1, 2, \cdots, p)$ denote p points in euclidean R_n. They determine the p-vector $\Xi=[X^{(1)}, \cdots, X^{(p)}]$. Its $N=\binom{n}{p}$ components are the minors of order p of the matrix $(x_{\pi\nu})$. If they are arranged in some definite order, Ξ can be interpreted as a point in R_N. Let $K^{(1)}, \cdots, K^{(p)}$ be symmetric convex bodies in R_n. If each $X^{(\pi)}$ ranges over $K^{(\pi)}$, Ξ will range over a point set in R_N whose convex hull $\mathrm{K}=[K^{(1)}, \cdots, K^{(p)}]$ is a symmetric convex body. Put $P=\binom{n-1}{p-1}$ and let c_1, c_2 denote positive constants which depend only on n and p. Let $V(K), \cdots$ denote the volume of $K, \cdots$ and put $S=V(K)\{\prod_{\pi=1}^{p} V(K^{(\pi)})\}^{-P/p}$. Mahler showed that S has no upper bound if $p>1$ and that (*) $S>c$, if there are only two distinct convex bodies among the $K^{(\pi)}$'s. The author proves (*) in the general case by symmetrizing each $K^{(\pi)}$ with respect to all the coordinate $(n-1)$-spaces. His proof uses the following lemma: Let $H^{(\pi)}$ be the convex body obtained from $K^{(\pi)}$ by Steiner symmetrization with respect to a given $(n-1)$-space through the origin; $\pi=1, 2, \cdots, p$. Then $V([H^{(1)}, \cdots, H^{(p)}]) \leqq c_2 V(K)$. — In the last section upper bounds for the successive minima of K and for their product are given without proofs. [Cf. Mahler, Proc. London Math. Soc. (3) **5** (1955), 358–379, 380–384; MR **17**, 589.] *P. Scherk.*

Lekkerkerker, C. G. On the volume of compound convex bodies. Nederl. Akad. Wetensch. Proc. Ser. A. **60**= Indag. Math. **19** (1957), 284–289.

Let $K^{(1)}, \cdots, K^{(p)}$ be p symmetric convex bodies in R_n where $1 \leqq p \leqq n-1$; let $N=\binom{n}{p}$ and $P=\binom{n-1}{p-1}$; and let $\mathrm{K}=[K^{(1)}, \cdots, K^{(p)}]$ be the compound convex body of $K^{(1)}, \cdots, K^{(p)}$ in R_N [K. Mahler, Proc. London Math. Soc. (3) **5** (1955), 358–379, 380–384; MR **17**, 589]. Put $Q=V(K)\prod_{\pi=1}^{p} V(K^{(\pi)})^{-P/p}$, where V denotes the volume. In his paper, Mahler stated the conjecture that $Q \geqq c$ where $c>0$ depends only on n and p. This conjecture was proved by C. A. Rogers [see paper reviewed above] by means of Steiner symmetrisation. The author, without knowledge of Rogers' work, gives a second and completely different proof which is based on Minkowski's theorem on the successive minima of a convex body in the lattice of points with integral coordinates. Let $m_1^{(\pi)}, \cdots, m_n^{(\pi)}$, for $\pi=1, \cdots, p$, be the successive minima of $K^{(\pi)}$, and let $A_1^{(\pi)}, \cdots, A_n^{(\pi)}$ be n independent lattice points at which these minima are attained. Then N systems of p suffixes

$$(\mu_{1i}, \cdots, \mu_{pi}) \quad (i=1, 2, \cdots, N)$$

can be constructed such that the compound points

$$\Xi^{(i)}=[A_{\mu_{1i}}^{(1)}, \cdots, A_{\mu_{pi}}^{(p)}]$$

in R_N are independent and that further

$$\prod_{i=1}^{N} (m_{\mu_{1i}}^{(1)} \cdots m_{\mu_{pi}}^{(p)}) \leqq \Big(\prod_{\pi=1}^{p}\prod_{\nu=1}^{n} m_\nu^{(\pi)}\Big)^{P/p}.$$

The 2^N points

$$\mp(m_{\mu_{1i}}^{(1)} \cdots m_{\mu_{pi}}^{(p)})^{-1}\Xi^{(i)} \quad (i=1, 2, \cdots, N)$$

form now the vertices of a generalised "octahedron" contained in K, and the lower bound for Q is a consequence of Minkowski's theorem. *K. Mahler* (Manchester).

See also: General Topology: de Groot.

Differential Geometry

Mirguet, Jean. Sur une généralisation de la stricte convexité. C. R. Acad. Sci. Paris **245** (1957), 402–404.

S: orthosurface (rondelle lipschitzienne ouverte) pour la direction Δ regardée comme verticale. M: point générique de S. Une demidroite MD issue de M est dite positive (négative) si l'intersection de S et du demi-plan vertical projetant MD est, sauf en M, au dessus (au dessous) de MD dans un voisinage de M. Une droite D_1MD_2 est dite positive (négative) si MD_1 et MD_2 sont toutes deux positives (négatives). Une paratingente de S en M est dite "libre" si elle est soit positive soit négative et si les paratingentes de S en M suffisamment voisines ont le même signe qu'elle. Théorème — Les propriétés suivantes sont équivalentes: (1) S admet en chaque point un plan entièrement constitué de paratingentes libres. (2) S est strictement convexe. Quelques renseignements sont donnés concernant les orthosurfaces S telles qu'en chaque point de S tout plan (totalement) paratingent inclue une paratingente non libre et une seule. *Chr. Pauc.*

Mirguet, Jean. Sur une opposition de courbures asymétrique déduite du paratingent libre. C. R. Acad. Sci. Paris **245** (1957), 488–490.

[Cf. analyse précédente.] Les orthosurfaces étudiées satisfont les conditions suivantes: (1) Si MD est une demi-tangente de S en M sans signe, alors l'intersection de S et du demi-plan vertical projetant MD consiste, dans un voisinage de M, en un segment rectiligne (porté par MD). (2) L'ensemble des paratingentes non libres en M n'a pas d'élément intérieur mais coupe chaque plan (totalement) paratingent en M en deux droites au moins. (3) Il existe une paratingente libre en M qui n'est pas supérieure [cf. Mirquet, Rev. Sci. **85** (1947), 67–72; MR **9**, 18]. Il est montré que ces surfaces S possèdent un plan tangent variant continuement et présentent une "opposition asymétrique ou symétrique des courbures".

Chr. Pauc (Nantes).

Savasta, Carmelo. Una notevole classe di geodetiche. Atti Soc. Peloritana Sci. Fis. Mat. Nat. **3** (1956–57), 343–346.

Si determinano le equazioni delle goedetiche dei cilindri (eliche) e si riconosce, a completamento della classica dimostrazione del teorema di Bertrand sulle eliche cilindriche, che esse coincidono con le curve il cui rapporto delle curvature sia costante, anche quando tale costante sia $\pm i$ (con $i=\sqrt{-1}$). *Riassunto dell'autore.*

Savasta, Carmelo. Sulle eliche cilindriche. Atti Soc. Peloritana Sci. Fis. Mat. Nat. **3** (1956–57), 339–342.

Si determinano le curve dello spazio ordinario complesso tali che il rapporto delle curvature sia costante.

Dal riassunto dell'autore.

Biran, Lutfi. Un problème élémentaire de géométrie différentielle. Rev. Fac. Sci. Univ. Istanbul. Sér. A. **21** (1956), 239–243 (1957). (Turkish summary)

Etant donnée une courbe gauche $[x]$, lieu d'un point x; soit $y^{(1)}$ le centre de la sphère osculatrice de $[x]$ au point x. Soit de même $y^{(2)}$, le centre de la sphère osculatrice à la courbe $[y^{(1)}]$ au point $y^{(1)}$, et ainsi de suite. L'auteur démontre alors le théorème suivant: Si le centre $y^{(n)}$ de la sphère osculatrice au point $y^{(n-1)}$ de la courbe $[y^{(n-1)}]$ tend vers une position limite y, lorsque $n\to\infty$, cette position limite reste invariée pour tous les autres points d'un arc s de $[x]$ qui contient x. *F. Şemin* (Istanbul).

Šmahel, Josef. Die angenäherte konforme Abbildung des Besselschen Ellipsoides. Apl. Mat. **2** (1957), 297–313. (Czech. Russian and German summaries)

The author studies the central projection of an ellipsoid of revolution onto a tangent plane, from a point P on the normal through the point of contact P_0, where $\frac{1}{2}PP_0$ is the reciprocal of the mean curvature at P_0. A table indicates that in the case of the earth's surface the map is practically conformal within a circle of radius 370 km about P_0; the distortion of angles remains less than 0.7".

F. A. Behrend (Victoria).

Nitsche, Johannes C. C. Elementary proof of Bernstein's theorem on minimal surfaces. Ann. of Math. (2) **66** (1957), 543–544.

The title of this paper refers to the theorem of S. Bernstein that a minimal surface $z(x, y)$ defined for all finite values of (x, y) is necessarily a plane. The original proof of Bernstein (1915) contained a gap (later to be removed, independently, by E. Hopf [Proc. Amer. Math. Soc. **1** (1950), 80–85; MR **12**, 13] and by E. J. Mickle [ibid. **1** (1950), 86–89; MR **12**, 13]). Alternative proofs have been given by T. Radó (1926), L. Bers [Ann. of Math. (2) **53** (1951), 364–386; MR **13**, 244], E. Heinz [Nachr. Akad. Wiss. Göttingen. Math.-Phys. Kl. IIa. **1952**, 51–56; MR **14**, 885], E. Hopf [J. Rational Mech. Anal. **2** (1953), 519–522, 801–802; MR **14**, 1119; **15**, 348] and K. Jörgens [Math. Ann. **127** (1954), 13–134; MR **15**, 961]. In this note the author presents in a single page a proof which is elementary and natural, and so simple it will hardly be improved. His result is striking in view of the widespread attention that the theorem has received and the relative difficulty of all previous demonstrations. The reviewer remarks that H. Jenkins, using the idea of the author's proof, has obtained generalizations of Bernstein's theorem which bring out clearly the structure of the non-linearity responsible for the particular behavior of the solutions.

In the second (and final) page of his paper, the author outlines the derivation of an a-priori estimate which includes the theorem of Bernstein as a special case.

R. Finn (Pasadena, Calif.).

Saban, Giacomo. Su particolari deformazioni infinitesime di superficie rigate. Atti Accad. Naz. Lincei. Rend. Cl. Sci. Fis. Mat. Nat. (8) **21** (1956), 274–278.

Dans cette note l'auteur développe les conséquences de deux équations scalaires (duales), relatives aux surfaces règlées, et conservant leur validité par les déformations infinitésimales de ces surfaces conservantl 'élément d'arc sphérique dual [Saban, Arch. Math. **7** (1956), 380–383; MR **18**, 923]. Il envisage, en particulier, les déformations conservant à la fois l'arc précédent et le paramètre de distribution, et parmi ces dernières celles qui n'altèrent pas l'indicatrice sphérique. Le cas où la déformation infinitésimale conserve le paramètre de distribution et la ligne de striction donne lieu à une caractérisation géométrique remarquable, et la considération des surfaces règlées fermées pour lesquelles la déformation infinitésimale conserve, ou bien le paramètre de distribution et la ligne de striction, ou bien le paramètre directeur, l'arc de l'indicatrice des génératrices et une directrice fermée, conduit à d'élégantes formules intégrales.

P. Vincensini (Marseille).

Poznyak, È. G. An example of a closed surface with singular point, having a countable fundamental system of infinitesimal deformations. Uspehi Mat. Nauk (N.S.) **12** (1957), no. 3(75), 363–367. (Russian)

The author constructs closed surfaces of revolution (with a singularity at one of the poles) possessing countably many independent infinitesimal deformations. At the regular pole the surfaces constructed are analytic, at the irregular pole they are given by some function like $r(z)=ze^{-1/z}$ (in cylindrical coordinates); in between they are of class C^2. The point is that all solutions of the differential equations for the Fourier coefficients of the infinitesimal deformation vector become zero at the irregular pole, so that any solution near the regular pole can be continued all the way to the irregular pole.

H. Samelson (Ann Arbor, Mich.).

Ryžkov, V. V. Affine tangential deformation of surfaces. Uspehi Mat. Nauk (N.S.) **12** (1957), no. 3(75), 195–200. (Russian)

Two surfaces V_n in affine E_n can be considered (locally) applicable of order k in the sense of E. Cartan [C. R. Con-

grès Internat. Math., Strasbourg, 1920, pp. 397–406] in three respects: as point manifolds, as tangential manifolds, or as point-tangential manifolds. The surfaces are given by $\bar{x}=\bar{x}(u^i)$, $\bar{y}=\bar{y}(u^i)$, $i=1, 2, \cdots, n$, the affine transformations by $\bar{\xi}\to A\bar{\xi}+\bar{b}$, $A(u^i)$ a matrix, $\bar{b}(u^i)$ a vector, and the equations expressing the applicability for neighborhoods of order k are set up for all three cases. Tangential application, not reducing to point-tangential application, imposes serious restrictions, there must be correspondence of conjugate systems; if a surface has an osculating plane of dimension $\geq\frac{1}{2}n(n+1)+2$, then its tangential applicabilities are either point-tangential ones ($k>1$) or reducible to them by a normalisation of A and $\bar{b}$. But from a tangential application of order k always follows a point application of order k in the same correspondence. For the case $k=2$ and point applicability we are referred to G. F. Laptev, Dokl. Akad. Nauk SSSR (N.S.) **58** (1947), 529–531 [MR **10**, 67]. *D. J. Struik.*

Geĭdel'man, R. M. Multidimensional systems R. Uspehi Mat. Nauk (N.S.) **12** (1957), no. 3(75), 285–290. (Russian)

The projective theory of line congruences in three space leads to the so-called R-congruences, for which the asymptotic lines on the focal surfaces correspond under all Laplace transformations. The present paper reports on generalizations of these congruences to k-conjugate systems in n-space, on which R. V. Smirnov wrote a previous report [Uspehi Mat. Nauk (N.S.) **4** (1949), no. 4(32), 162–163; MR **11**, 397] and a paper [Dokl. Akad. Nauk SSSR (N.S.) **71** (1950), 437–439; MR **11**, 616]. An essential role is played by the asymptotic lines and the asymptotic cones formed by their tangents at each point; on these cones some theorems are given. [See also Geĭdel'man, Mat. Sb. N.S. **34(76)** (1954), 499–524; **36(78)** (1955), 209–232; MR **16**, 168, 1149]. *D. J. Struik.*

Marcus, F. Sur les surfaces de troisième espèce de Terracini. Czechoslovak Math. J. **6(81)** (1956), 559–562. (Russian summary)

L'auteur retrouve et complète certains résultats de O. Rozet et N. Legrain-Pissard [Bull. Soc. Roy. Sci. Liège **23** (1954), 280–296; MR **16**, 513] sur les surfaces non réglées de troisième espèce de Terracini dont les asymptotiques appartiennent à des complexes linéaires [G. Fubini et E. Čech; Geometria differenziale proiettiva, Bologna, 1920, T. I. N. Zanichelli]. Il envisage, en particulier, le résultat suivant lequel une surface de troisième espèce de Terracini ne peut pas être transformée en une quadrique par une congruence W.

S'appuyant sur un résultat d'O. Mayer il montre que les surfaces de troisième espèce de Terracini sont caractérisées, dans la famille générale des surfaces de Terracini, par la propriété d'être des surfaces minima projectives. Il tire alors sa conclusion du fait que ces dernières surfaces sont un cas limite de surfaces de Tzitzéica-Wilczynski, ne pouvant pas être transformées en quadriques par congruences W. *P. Vincensini* (Marseille).

See also: Functions of Real Variables: de Rham. **Manifolds, Connections:** Gołąb.

Manifolds, Connections

★ **Gołąb, Stanisław. Rachunek tensorowy. [Tensor calculus.]** Biblioteka Matematyczna, Tom 11. Państwowe Wydawnictwo Naukowe, Warsaw, 1956. 309 pp. zł. 30.70.

This is a careful book, in the classical style and the usual best traditions of the Polish school, on the Tensor Calculus, written from a geometrical point of view, and intended for students of Physics and Engineering as well as those of Mathematics. By an explicitly stated preference, the author follows the methods of Schouten, tensors appear as things with many indices; the concepts of dual spaces, multilinear maps, exterior multiplication, tensor products and tensor algebras, and in general, the ideas of E. Cartan and the French school, do not appear in the book.

There are the usual two parts, the algebra of tensors and tensor analysis. Ch. 1 contains a semi-historical account of topological, metric and Euclidean spaces, followed by a discussion of the geometrical problems of coordinatizations, Klein's Erlangen program, geometries based on groups and quasigroups, and the structure of various subgroups of the affine group. Ch. 2 introduces vectors, vector spaces and operations on vectors; this is followed by Schouten's theory of a geometrical object (although no credit is given to Grassmann who originated the subject). Ch. 3 introduces p-vectors, tensors and tensor densities, and operations on these; this is illustrated by many geometrical examples. Ch. 4 completes Part 1 with a discussion of some special topics (dyadics, systems of holonomy, hessians of vector fields, etc.) and a preparation for Part 2.

This begins with a long chapter on parallelisms, covariant differentiation, absolute derivatives, Crhistoffel symbols, geodesics in Riemannian spaces and geodesic coordinates. Ch. 6 introduces and discusses in detail the curvature and torsion tensors. Ch. 7 is devoted to a detailed treatment of the geometry of Riemannian spaces. Chapters 8 and 9 take up selected special topics, such as the spaces of Weyl, differential invariants and the theorems of Gauss, Stokes and Green. The last chapter contains some applications of tensors: the equations and properties of geodesics, some aspects of embedding problems, and a brief but excellent exposition of classical differential geometry.

A five-page collection of symbols, definitions and formulas, a bibliography of 117 references and an index complete the book. *Z. A. Melzak* (Montreal, P.Q.).

Nagarathnamma, H. S. (Miss) Properties of the intrinsic derivatives of the first and higher orders of the unit normal vector for a curve in a Riemannian space. Proc. Nat. Inst. Sci. India. Part A. **23** (1957), 40–49.

C is a curve in V_n, which itself is a hypersurface of a Riemannian V_{n+1}. Relations are found involving the curvature and torsion of C in V_{n+1} and the first and higher order intrinsic derivatives of the unit principal normal of C in V_{n+1}. A number of theorems are proved; the following is typical: "If C is an asymptotic line of V_n, the ratio of the torsion of C in V_n to that of C in V_{n+1} is numerically equal to the ratio of the cosine of the angle between the principal normal to C in V_n and the first binormal to C in V_{n+1}, to the cosine of the angle between the principal normal to C in V_{n+1} and the first binormal to C in V_n". *T. J. Willmore* (Liverpool).

Allamigeon, André Claude. Espaces harmoniques décomposables. C. R. Acad. Sci. Paris **245** (1957), 1498–1500.

Lichnerowicz proved [Bull. Soc. Math. France **72** (1944), 146–168; MR **7**, 80] that a decomposable harmonic Riemannian space with definite metric is locally euclidean. In this paper decomposable harmonic spaces with indefinite metrics are considered. It is found that the func-

tion $\rho(\Omega)$ for such spaces must assume the special form $\rho=e^{k\Omega}$. It is proved that if V_μ, W_ν are two Riemannian spaces whose Riemannian product H_n is harmonic, then V_μ, W_ν are harmonic, and moreover, H_n, V_μ, W_ν are harmonic spaces whose ρ-functions assume the form $\rho=e^{k\Omega}$ with the same value of the parameter k.

Various conditions are imposed upon V_μ, H_n, which imply that H_n is simply harmonic. In particular it is proved that a harmonic recurrent space is simply harmonic — a result previously obtained by A. G. Walker [Proc. London Math. Soc. (2) **52** (1950), 36–64, p. 62; MR **12**, 283]. *T. J. Willmore* (Liverpool).

Su, Buchin. Koschmieder invariant and the associate differential equation of a minimal hypersurface in a regular Cartan space. Math. Nachr. **16** (1957), 117–129.

This is a sequal to another paper [Acta Math. **85** (1951), 99–116; MR **12**, 749] by the author on the infinitesimal deformation of a hypersurface in a regular Cartan space. Whereas in that paper the author considered infinitesimal deformations of the form $\delta x=\xi(x)\delta t$, in this case the corresponding deformation is $\delta x=\xi(x, p)\delta t$, where p is the covariant tangential vector. The paper follows closely the treatment of Cartan spaces given by Berwald [Acta Math. **71** (1939), 191–248; MR **1**, 177]. A minimal hypersurface is characterized by the vanishing of the invariant known as the mean curvature. The problem treated here is that of finding the condition in order that a minimal hypersurface shall remain minimal under the deformation $\xi(x, p)\delta t$. It is shown that the differential equation thus arrived at involves an invariant called the Koschmieder invariant in Berwald's work. *E. T. Davies.*

Akbar-Zadeh, Hassan. Sur une connexion euclidienne d'espace d'éléments linéaires. C. R. Acad. Sci. Paris **245** (1957), 26–28.

The author considers an n-dimensional variety V, its bundle of tangent vectors V and its (oriented) tangent directions W. W is said to have a finslerian structure if there exists a map assigning to each $z\in W$ a symmetric, positive definite, covariant tensor g_{ij} at $p(t)$, i.e., an element of $T^*_{p(t)}\otimes T^*_{p(t)}$, where $p:W\to V$ is the bundle projection. Using these concepts, he gives a new characterization of a finsler connection and a Cartan connection on a finsler space. Also he gives conditions that $g_{ij}=\partial_{ij}{}^2F$, where $F(x, X)=\frac{1}{2}g_{ij}(x, X)X^iX^j$, $X=(X^1, \cdots, X^n)$ a tangent vector to V. *W. M. Boothby* (Evanston, Ill.).

Complex Manifolds

Bremermann, H. J. Construction of the envelopes of holomorphy of arbitrary domains. Rev. Mat. Hisp.-Amer. (4) **17** (1957), 175–200.

Let D be a domain over a complex Banach space. The author defines two operations: 1) the c-process (continuation process), acting on D by identifying certain points of D, and by replacing a subset S contained in a schlicht subset $U\subset D$ by a larger subset S^*; 2) the i-process (identifying process) operating on D by a suitable identification of certain subsets of D. If D and its envelope of holomorphy $E(D)$ are schlicht, then the use of the i-process can be skipped, and the c-process can be simplified. Let $c(\cup)$ denote this simplified c-process. The author proves that, if D and $E(D)$ are schlicht domains, then, by iteration of $c(\cup)$, a sequence of domains D_ν arises, which converges always. Furthermore the limit is unique. If it is a pseudo-convex domain, it is equal to the pseudo-convex envelope $C(D)$ of D (i.e. the intersection of all pseudo-convex domains containing D), and every function holomorphic in D is holomorphic in $C(D)$. Moreover, it is shown that, if D is contained in the complex number space C^n, it is possible to choose the subsets $S_{\nu-1}\subset D_\nu$ and to iterate the $c(\cup)$-process in such a way that the sequence $\{D_\nu\}$ converges towards a pseudo-convex domain. So, in view of a theorem of K. Oka [Tôhoku Math. J. **49** (1942), 15–52; Jap. J. Math. **23** (1953), 97–155; MR **7**, 290; **17**, 82–83], the author is able to conclude that, if D and $E(D)$ are schlicht, and if $D\subset C^n$, then $E(D)=C(D)=\lim D_\nu$. If $E(D)$ is not schlicht, then, even by using both the c- and i-processes, the question is left open, whether, by a suitable choice of the subsets $S_{\nu-1}$, the iteration of the c-i-process converges to a pseudo-convex limit. This method works even for holomorphic functionals in complex Banach spaces of infinite dimensions. But in that case — even if D and $E(D)$ are schlicht — the limit of the iteration of $c(\cup)$ is not proved — but only conjectured — to be the envelope of holomorphy of D. In order to achieve the above results, an amount of background material is reviewed. The equivalence between the author's [Trans. Amer. Math. Soc. **82** (1956), 17–51; MR **18**, 28] and Oka's [see the second paper cited above] definitions of pseudo-convexity is proved. *E. Vesentini.*

Tsukamoto, Yôtarô. On Kählerian manifolds with positive holomorphic sectional curvature. Proc. Japan Acad. **33** (1957), 333–335.

Three theorems are proved which are generalizations of known theorems for Riemannian manifolds: (1) A complete Kählerian manifold whose holomorphic sectional curvature is greater than or equal to $e>0$ is compact and has diameter $\leq\pi/\sqrt{e}$. (2) A complete Kählerian manifold with positive holomorphic sectional curvature contains no closed geodesic of minimum type and hence, (3), is simply connected. *C. B. Allendoerfer.*

Algebraic Geometry

Spampinato, Nicolò. La varietà dell' S_{10} determinata da una superficie algebrica dell' S_3 complesso prolungata in un'algebra del 5° ordine. Giorn. Mat. Battaglini (5) **5(85)** (1957), 31–40.

Rollero, Aldo. Un'osservazione sulle cubiche piane cuspidate. Atti Accad. Ligure **13** (1957), 142–144.

Chow, Wei-Liang. On the principle of degeneration in algebraic geometry. Ann. of Math. (2) **66** (1957), 70–79.

Zariski proved the principle of degeneration in abstract algebraic geometry in his long and difficult paper [Mem. Amer. Math. Soc., no. **5** (1951); MR **12**, 853]. Here the author generalizes and proves it, the proof being rather short.

Let R be a noetherian ring, K be its quotient field and $\mathfrak{p}$ be a prime ideal in R. Let Z be a positive cycle in the projective space S^n. Let $F(U)$ be the Chow-form (associated-form, zugeordnete Form) of Z and $\bar{F}(U)$ be the form obtained from $F(U)$ by reducing coefficients modulo $\mathfrak{p}$. Then $\bar{F}(U)$ is the Chow-form of a positive cycle $\bar{Z}$. The generalized principle of degeneration asserts that "if Z is connected, $\bar{Z}$ is connected (in the sense of Zariski topology)".

The question is reduced to the case when K is a complete real discrete valuation field and $\mathfrak{p}$ is its maximal prime ideal. After this reduction, the author shows that "when $F(U)$ is rational over K and when Z is K-connected, then $\bar{Z}$ is $R/\mathfrak{p}$-connected". The main idea is to regard Hensel's lemma as the principle of degeneration on the projective straight line. Then he generalizes Hensel's lemma to a splitting polynomial of several variables (i.e., a polynomial which is the product of linear polynomials over some extension of the field of reference), which gives the principle of degeneration for 0-cycles on projective spaces. After proving one crucial lemma about coefficients which enter in the power series development given by the generalized Hensel's lemma, the author applies induction on the dimension of Z, which settles this principle relative to fields. The above mentioned absolute principle of degeneration follows immediately from this. *T. Matsusaka* (Evanston, Ill.).

Levi, Beppo. Singular points and varieties over algebraic and analytic varieties. I, II. Math. Notae **15** (1955), 1–62, 73–129. (Spanish)

Quoting from the text, we have this statement by the author: "I believe that the totality of these last considerations, while it must serve as a proof of the effective resolvability of the singularities of an algebraic hypersurface of any order and dimension, underscores the practical difficulties encountered so far in previous endeavours on this argument. Moreover they may show an effective difficulty in looking for a precise result of an absolutely general character, probably insuperable because the number of branches that the problem presents tends to infinity with increasing dimension and degree."

The reviewer regrets to say that a careful reading of these two papers has failed to elicit any proof of the reduction of singularities. *E. Lluis* (Mexico, D.F.).

Hironaka, Heisuke. On the arithmetic genera and the effective genera of algebraic curves. Mem. Coll. Sci. Univ. Kyoto. Ser. A. Math. **30** (1957), 177–195.

Curve here means positive one-cycle in projective space having no multiple components. The main result, which relates the arithmetic genus of a curve to the (effective) genera of its components and the orders of singularity of the singular points of the curve (including the intersections of two or more components) had previously been proved by the reviewer [Ann. of Math. (2) **56** (1952), 169–191; MR **14**, 80]. For a curve on a nonsingular surface, the order of singularity of a point is shown to be the sum of its orders of singularity for the various components containing it plus the sum of the various local intersection multiplicities for pairs of such components, and the classical formula expressing the order of singularity in terms of the multiplicities of neighboring points (defined by repeated quadratic transformations) is proved generally. Finally, the behavior of a curve under specialization is studied: assuming that the specialization introduces no multiple components, a formula is given for the (possible) rise in arithmetic genus and it is shown that the sum of the effective genera of the components at worst falls.

M. Rosenlicht (Evanston, Ill.).

Thalberg, Olaf M. 'Conic involutions' with a coincident curve of order $3n+4$. Avh. Norske Vid. Akad. Oslo. I. **1956**, no. 1, 20 pp.

Dans des travaux antérieurs [même Avh. **1947**, no. 1; **1952**, no. 1; **1953**, no. 1; MR **9**, 464; **16**, 65, 740] l'auteur a étudié les courbes de coïncidence des espèces indiquées dans les involutions coniques du plan, c'est-à-dire pour lesquelles deux points conjugués quelconques sont situés sur une même conique K passant par quatre points fixes (les pôles) a, b, c, d du plan. Il envisage, dans le travail actuel, d'abord le cas où la courbe de coïncidence est une cubique passant par les quatre pôles, puis celui où cette même courbe est d'ordre $4n+3$ et passe par a^{2n+1}, b^{2n+1}, c^{2n+1}, d^{2n+1}, et l'étude minutieuse qu'il fait des nombreuses et diverses circonstances qui peuvent se présenter, conduit à un ensemble de résultats formulés dans une suite de théorèmes d'un grand intérêt géométrique.

P. Vincensini (Marseille).

See also: Fields, Rings: Zariski and Samuel; Rees.

NUMERICAL ANALYSIS

Numerical Methods

Mihlin, S. G. Some sufficient conditions for convergence of Galerkin's method. Leningrad. Gos. Univ. Uč. Zap. **135**. Ser. Mat. Nauk. **21** (1950), 3–23. (Russian)

Let A be a linear operator defined on a linear subset M of a separable Hilbert space H, M being dense in H. Given f in H, Galerkin's method for the solution of the equation $Au-f=0$ consists in the following: choose a complete sequence φ_n in M, and construct the "approximations" $u_n=\sum_{k=1}^{n} a_{kn}\varphi_k$ to the unknown u, where the constants a_{kn} are determined by the system of linear equations $\sum_{k=1}^{n}(A\varphi_k, \varphi_j)a_{kn}=(f, \varphi_j)$ $(j=1, 2, \cdots, n)$. If the equation $Au-f=0$ happens to be the Euler-Lagrange equation for some variational problem, then Galerkin's method coincides with the method of Ritz, the convergence of which has been studied extensively, but in other cases little seems to be known about the convergence of Galerkin's method. G. I. Petrov [Ž. Prikl. Mat. Meh. (N.S.) **4** (1940), no. 3, 3–12] and M. B. Keldiš [Izv. Akad. Nauk SSSR. Ser. Mat. **6** (1942), 309–330; MR **5**, 7] have considered the convergence of Galerkin's method for special operators A. In the present paper the author gives a more general criterion for the convergence of Galerkin's method and applies it to various problems of mathematical physics, in particular to the problems considered earlier by Keldiš. [For another presentation see ch. 5 of Mihlin, Direct methods in mathematical physics, Gostehizdat, Moscow-Leningrad, 1950; MR **16**, 41.]

J. B. Diaz (College Park, Md.).

Ostrowski, A. M. A method of speeding up iterations with super-linear convergence. J. Math. Mech. **7** (1958), 117–120.

In a previous paper [Z. Angew. Math. Phys. **7** (1956), 218–229; MR **18**, 71] the author examined the rate of convergence of the Steffensen and Householder methods of speeding up an iteration $z_{k+1}=\varphi(z_k)$. The present paper is based on the fact that if this iteration converges superlinearly, the convergence with the Steffensen iterating function need not be better or even as good as the iteration by the iterating function $\varphi(\varphi(z))$. The author then shows that in this case another improvement of the convergence can be obtained by using the iterating

function

$$\Psi(z)=\varphi(\varphi(z))-\{[\varphi(z)-\varphi(\varphi(z))]^{n+1}/[z-\varphi(z)]^n\};$$

here the integer n (>1) is determined by $\varphi(z)-\zeta=\alpha(z-\zeta)^n(1+o(1))$ for $z\to\zeta$ where ζ is the fixpoint of the iteration and $\alpha\neq 0, \infty$. With some slight additional assumptions the author obtains a precise statement on the rate of convergence of the iteration $z_{k+1}=\Psi(z_k)$.

W. C. Rheinboldt (Washington, D.C.).

Stegun, Irene A.; and Abramowitz, Milton. Pitfalls in computation. J. Soc. Indust. Appl. Math. **4** (1956), 207–219.

In addition to elementary pitfalls, such as occur in computing trigonometric functions of large arguments or square roots of real and complex numbers, the authors discuss more sophisticated errors occurring in series calculations, numerical differentiation and numerical integration. The discussion is particularly addressed to numerical analysts using automatic computers.

W. F. Freiberger (Providence, R.I.).

Allred, John C.; and Newhouse, Albert. Applications of the Monte Carlo method to architectural acoustics. J. Acoust. Soc. Amer. **30** (1958), 1–3.

The authors have three aims in this paper: (1) to investigate the application of Monte Carlo machine methods to the design of auditoria; (2) to calculate the relative collision probabilities of sound waves (isotropic source) with various walls for rectangular parallelepipeds (note: the authors assume a ray structure of the sound wave and ignore diffraction effects); and (3) to estimate the average distance between points of collision of a sound wave (ray) with the walls of the enclosure, i.e. estimate the mean free path. A table of the results of machine runs for the mean free path for fifteen room shapes is given and the results are compared with those obtained from a conventional approximation. This paper gives an interesting example of the use of a Monte Carlo approach for studying a physical phenomenon. However, the paper does not dwell on the statistical implications underlying the process. A table is given for the distribution of wall collision frequencies for a 10′:10′:10′ room. The statistical analysis of certain of the results is open to question, especially since for any fixed number of wall collisions the deviations of the observed frequencies from the expected frequencies have two, and not three, degrees of freedom.

M. Muller (Princeton, N.J.).

Unbehauen, Rolf. Ein numerisches Verfahren zur Ermittlung einer Zweipolfunktion, welche in einem Intervall reeller Frequenzen eine gegebene komplexe Funktion approximiert. Arch. Elek. Übertr. **11** (1957), 440–448.

A method is given such that, by numerical processes, a two-pole function is found which approximates a given complex function in an interval of real frequencies. After transformation of the variables, the approximation problem is first worked out by means of polynomials. Then general rational functions are obtained which approximate the polynomials found in the transformed frequency range. An example which illustrates the method is worked out.

E. Frank (Chicago, Ill.).

Davison, B. Multilayer problems in the spherical harmonics method. Canad. J. Phys. **35** (1957), 55–63.

The spherical harmonics method for solving neutron transport problems in reactor theory was described for a single medium by Davison [Neutron transport theory, Oxford Univ. Press, London, 1956]. In the present paper the problem is the extension of the one-velocity theory to multilayer problems. The most laborious part of the calculation is to determine certain constants of integration for each consecutive layer from the previous one.

Under certain physical assumptions, the author reduces the above calculation to the inversion of a matrix of values of certain special functions too complicated to reproduce. The inversion is reviewed in the slab case, and, in the case of spherical symmetry, it is presented as a new result.

Some critical size problems are solved in illustration.

G. E. Forsythe (Stanford, Calif.).

Wagner, Harvey M. A comparison of the original and revised simplex methods. Operations Res. **5** (1957), 361–369.

An elementary deduction of the revised simplex method and a demonstration of its superiority over the original one.

W. F. Freiberger (Providence, R.I.).

Gleyzal, A. An algorithm for solving the transportation problem. J. Res. Nat. Bur. Standards **54** (1955), 213–216.

This article describes a new computational scheme for solving the transportation problem in which combinatorial ideas, rather than the theory of linear inequalities, play the major role. *From the author's summary.*

Fulkerson, D. R.; and Dantzig, G. B. Computation of maximal flows in networks. Naval Res. Logist. Quart. **2** (1955), 277–283 (1956).

The author exhibits a simple computational method for determining a maximal flow in a graph between two given vertices, where a capacity is assigned to each edge.

W. T. Tutte (Toronto, Ont.).

Ostrowski, Alexander. Über näherungsweise Auflösung von Systemen homogener linearer Gleichungen. Z. Angew. Math. Phys. **8** (1957), 280–285.

Bei der praktischen Auflösung eines homogenen linearen Gleichungssystems (*) $S\zeta'=0$ (S singuläre quadratische Matrix, ζ Zeilenvektor) können Unsicherheiten entstehen, wenn die Elemente von S nur näherungsweise berechnet wurden und die Determinante mit diesen Näherungselementen nicht verschwindet. Dies ist z.B. der Fall, wenn man Eigenvektoren einer Matrix berechnen will und eine Näherung für den Eigenwert eingesetzt hat. Es wird eine allgemeine Methode angegeben, mit welcher man eine Lösung π von (*) angenähert ermitteln kann. Sei $S+A$ ($A=(a_{\mu\nu})$) die gestörte Matrix mit nicht verschwindender Determinante, so löse man das inhomogene Gleichungssystem $(S+A)\xi'=\eta'$ mit geeignetem Vektor η. Es ist $\xi/|\xi|=\pi+O(\sqrt{\sum_{\mu,\nu}|a_{\mu\nu}|^2})$, wobei π jedoch davon abhängen kann, in welcher Weise die $a_{\mu\nu}$ gegen Null streben. Sind die $a_{\mu\nu}$ analytische Funktionen eines Parameters t, regulär bei $t=0$, so gilt $\varepsilon^j\pi=\tau+O(t)$ mit einer natürlichen Zahl j, $\varepsilon=t/|t|$ und einem konstanten Lösungsvektor τ von (*). Im oben erwähnten Fall der Eigenwertaufgabe wird $A=t\cdot E$ (E Einheitsmatrix, t Fehler des Eigenwertes). Bei reeller symmetrischer Matrix S und reellem t gilt dann z.B. $\xi/|\xi|=\tau \operatorname{sgn} t+O(t)$.

J. Schröder (Hamburg).

Madić, Petar B. Error domain in the solution of systems of linear algebraic equations. Bull. Soc. Math. Phys. Serbie **8** (1956), 191–194. (Serbo-Croatian. English summary)

If the vector $\mathbf{X}_p$ represents the approximate solution of the system $\mathbf{A}\cdot\mathbf{X}=\mathbf{b}$, then the sum of the squares of the errors is $(\mathbf{X}-\mathbf{X}_p)^t\cdot\mathbf{A}^t\cdot\mathbf{A}\cdot(\mathbf{X}-\mathbf{X}_p)=c$, where c is a positive constant and t indicates transpose. The domain of errors in the solutions is therefore given by:

$$|c\lambda^{-1}|\leq|\mathbf{X}-\mathbf{X}_p|^2\leq|c\mu^{-1}|,$$

where λ and μ are respectively the greatest and the least eigenvalue of the matrix $\mathbf{A}^t\cdot\mathbf{A}$.

From the author's summary.

Petrov, G. I. Estimation of accuracy in the approximate calculation of an eigenvalue by the method of Galerkin. Prikl. Mat. Meh. **21** (1957), 184–188. (Russian)

Galerkin's method (using a complete set of functions) leads to a system of infinitely many equations in infinitely many unknowns; this system is then treated by the method of segments. Estimates, in terms of the coefficients of the infinite system, are given for the accuracy of approximate eigenvalues obtained in this way.

R. C. T. Smith (Armidale).

Valat, Jean. Influence de l'amortissement sur un simulateur électromécanique d'une équation de Hill. C.R. Acad. Sci. Paris **244** (1957), 3017–3020.

The author considers the differential equation: $y''+\alpha y'+(\lambda+\tilde{\eta})y=0$, where α and λ are constants, whilst $\tilde{\eta}$ is a periodic function of frequency $f=1/T$, whose amplitude changes abruptly between $+\eta$, $-\eta$, $+\eta$, and so on. By a well-known procedure, the eigenvalues for a solution $y=A\phi(t)\exp(\mu t)$ are found from a matrix equation of known matrix elements. The author calculates zones of stability (μ imaginary) or instability (μ real) in a plane of coordinates $|\eta|T^2/4\pi^2$ and $(\lambda-\alpha^2/4)/4\pi^2$, and also calculates curves of fixed values of αT. These show the degree of instability.

M. J. O. Strutt (Zürich).

Gillies, D. B.; and Hunt, P. M. A solution of the flutter determinant on a general purpose electronic digital computer. Aero. Quart. **8** (1957), 185–203.

The authors study mechanical systems with N degrees of freedom arising in aerodynamics, and seek the complex frequencies associated with the various modes. In particular, critical frequencies (pure imaginary) are sought as functions of a parameter related to the aircraft speed. The problem reduces to that of evaluating the characteristic equation of various matrices (essentially by Leppert's method) and finding the zeros of this equation (essentially by Lin's method). For exploratory work, Routh's test functions are computed. The authors describe a flexible, fast program for a digital computer. Two appendices give a careful discussion of errors.

M. A. Hyman (Yorktown Heights, N.Y.).

Kantorovitch, L. V. On some further applications of the Newton approximation method. Vestnik Leningrad. Univ. Ser. Mat. Meh. Astr. **12** (1957), no. 7, 68–103. (Russian. English summary)

In this paper results of two earlier papers [Dokl. Akad. Nauk SSSR (N.S.) **76** (1951), 17–20; **80** (1951), 849–852; MR **12**, 835; **13**, 469] on the extension of Newton's method for functional equations on spaces, valued by the elements of a partially ordered space, are specialized to the case of normed spaces. The main theorem of the eight given states that there exists a unique solution x^* of the functional equation $P(x)=0$, where P is an operator twice continuously differentiable, which maps the normed space X onto Y, if the following conditions are satisfied: 1) There exists a real valued majorizing function $Q(x)$ on an interval (z_0, z') (i.e., $\|P(x_0)\|\leq Q(z_0)$ and $\|P'(x)\|\leq Q'(z)$ if $\|x-x_0\|\leq z-z_0\leq z'-z_0$) for which the relation $Q(z)=0$ has real roots z_1, $z_2(z_0\leq z_1\leq z_2\leq z')$; 2) There exists an inverse operator $\Gamma_0=-[P'(x_0)]^{-1}, B=-[Q'(z_0)]^{-1}>0$; 3) $\|\Gamma^0P(x_0)\|\leq BQ(z_0)$; 4) $\|\Gamma^0P''(x)\|\leq BQ''(z)$ for $\|x-x_0\|\leq z-z_0\leq z'-z_0$; The solution x^* is bounded by $\|x^*-x_0\|\leq z_1-z_0$ and furthermore it is unique in $\|x-x_0\|\leq z_2-z_0$. The approximations x_n obtained by the Newton method $(x_{n+1}=x_n-[P'(x_n)]^{-1}P(x_n))$ and its modification $(x_{n+1}=x_n-[P'(x_0)]^{-1}P(x_n))$ converge to x^*.

The results are extended also to functional equations involving a parameter. The theory is applied to a few examples in the fields of nonlinear integral equations and ordinary first order differential equations to derive error bounds for approximate solutions.

U. W. Hochstrasser (Lawrence, Kans.).

Chao, F. H. Newton's method for finding complex roots. Acta Math. Sinica **5** (1955), 137–147. (Chinese. English summary)

Polak, A. I. On sufficient and necessary conditions for complete approximative solvability of equations of a very general nature. Dokl. Akad. Nauk SSSR (N.S.) **112** (1957), 587–590. (Russian)

Let L be a continuous linear map of a Banach space R into a Banach space R'. A solution of the equation $L(x)=y$ is desired, which lies in a subset G of R, as a limit of a suitable sequence (x_n) of solutions of $L_n(x)=y_n$, where $y_n\to y$ and the sequence of linear operators (L_n) converges to L uniformly on G. It is seen that a necessary and sufficient condition for this is that the set (L_n) be uniformly open in the sense that for each $\varepsilon>0$ there is a $\delta>0$ such that the image of the ε-sphere of R under each L_n covers the δ-sphere of R'. A similar result for non-linear maps and an application to non-linear integral equations are given.

R. G. Bartle (Urbana, Ill.).

Thacher, Henry C., Jr. Optimum quadrature formulas in s dimensions. Math. Tables Aids Comput. **11** (1957), 189–194.

Let a numerical integration formula be given by

$$(1)\qquad \int_S \phi(x)dx=\sum_{i=1}^{m} c_i\phi(x_i)-R\phi,$$

where $R\phi$ is the error, S is an s-dimensional region, x a point in E_s and dx a volume element. If one requires that $R\phi=0$ for all monomials ϕ of degree at most n then necessarily for $0\leq m_1+\cdots+m_s\leq n$

$$(2)\qquad \sum_{i=1}^{m} c_i\prod_{j=1}^{s}(x_i^{(j)})^{m_j}=I_{m_1\cdots m_s},$$

where the term on the right is the integral over S of $x_1^{m_1}\cdots x_s^{m_s}$. Now write $G=[c_i{}^i\delta_{h_i}]$ and $X^{(j)}=[x_i^{(j)}\delta_{h_i}]$ so that G on $X^{(j)}$ are $m\times m$ matrices. Then (2) may be written

$$(3)\qquad \operatorname{tr}\Big\{G\prod_{j=1}^{s}(X^{(j)})^{m_j}G\Big\}=I_{m_1\cdots m_s}.$$

Using (3), the author derives formulas for the hypercube for $n=2$ and $n=3$, the former involving $s+1$ points at

least and the latter $2s$ points. It is suggested that the same sort of method applies to multidimensional analogues of the Laguerre and Hermite formulas.

The author should publish the proof that a five point third degree formula for the 3-cube is impossible as claimed, since the reviewer and Stroud have shown that a 5-point 3rd degree is possible for the tetrahedron.

P. C. Hammer (Madison, Wis.).

Moran, P. A. P. Addendum to the paper 'Numerical evaluation of a class of integrals'. Proc. Cambridge Philos. Soc. **53** (1957), 928.

Remark relating the paper in question [same Proc. **52** (1956), 230–233; MR **17**, 901] to that of H. Ruben, Biometrika **41** (1954), 200–227 [MR **16**, 153].

Hsu, Lee-tsch C. A general approximation method of evaluating multiple integrals. Tôhoku Math. J. (2) **9** (1957), 45–55.

"The object of this paper is to investigate a general method concerning the approximate evaluation of multiple integrals with periodic continuous functions as integrands." Particular attention is paid to the case of double integrals and triple integrals which are taken over circular regions and spherical domains respectively.

P. Civin (Eugene, Ore.).

Wilf, Herbert S. An open formula for the numerical integration of first order differential equations. Math. Tables Aids Comput. **11** (1957), 201–203.

Der Verf. leitet Formeln zur numerischen Integration von Anfangswertaufgaben mit der Differentialgleichung $y'=f(x, y)$ her. Die Werte $y_0'=f_0$, $y_1'=f_1$, $\cdots$, $y_{n-1}'=f_{n-1}$ ($f_i=f(x_i, y_i)$, $x_i=x_0+ih$, $y_i=y(x_i)$) werden als Summe von Linearkombinationen der y_0, y_1, $\cdots$, y_{n-1} und Restgliedern $O(h^{n-1}y^{(n)}(\xi))$ dargestellt und die y_i dann aus diesen Gleichungen z.T. eliminiert. Für $n=4$ ergibt sich z.B. nach einer weiteren Umformung

$$y_1-y_0=\frac{h}{12}[5f_0+8f_1-f(x_2,y_2^*)]+h^4\cdots,$$

$$y_2^*=5y_0-4y_1+2h[f_0+2f_1].$$

Am Beispiel $y'=1+y, y(0)=2$ werden die mit dieser Formel erhaltenen Ergebnisse mit denen verglichen, welche eine entsprechende Runge-Kutta-Formel liefert.

J. Schröder (Hamburg).

Fujita, Hiroichi. Oscillation represented by the third order differential equations. I. Proc. Fac. Engrg. Keio Univ. **8** (1955), 61–67.

The solutions of the circuit equation for the grid voltage of the Hartley and Colpitts oscillators with neglected grid current are described when the nonlinear characteristics of the vacuum tube are replaced by linear approximations. To accomplish this, a geometric description of trajectories of third order linear differential equations with constant coefficients is given in the phase space $(x, dx/dt, d^2x/dt^2)$.

W. R. Utz (Columbia, Mo.).

★ **Liniger, Werner. Zur Stabilität der numerischen Integrationsmethoden für Differentialgleichungen.** Université de Lausanne, 1957. 95 pp.

Douglas, Jim, Jr. The solution of the diffusion equation by a high order correct difference equation. J. Math. Phys. **35** (1956), 145–151.

The author proposes a six-point implicit difference scheme for the numerical solution of $u_t=u_{xx}$. The usual six-point implicit scheme [Crank and Nicholson, Proc. Cambridge Philos. Soc. **43** (1947), 50–67; MR **8**, 409] has a truncation error of order Δt. The author shows that the new scheme has an error of $(\Delta t)^2$, assuming that u is strongly continuous. The extra work (over Crank, Nicolson) is slight. Because the error is smaller, it will be possible when employing the new method to use relatively large Δx, Δt (i.e., cover a region with fewer mesh-points), an important practical result.

M. A. Hyman.

Douglas, Jim, Jr. A note on the alternating direction implicit method for the numerical solution of heat flow problems. Proc. Amer. Math. Soc. **8** (1957), 409–412.

An effective numerical method for integrating $u_{xx}+u_{yy}=u_t$ or $u_{xx}+u_{yy}=0$ is the "alternating-direction" technique [Peaceman and Rachford, J. Soc. Indust. Appl. Math. **3** (1955), 28–41; MR **17**, 196; Douglas, ibid. **3** (1955), 42–65; MR **17**, 196]. As originally presented, the method applies to rectangular regions. Douglas here extends it to regions with polygonal boundaries, each segment of the boundary being parallel to one of the coordinate axes.

M. A. Hyman (Yorktown Heights, N.Y.).

Hajdin, Nikola. Ein Verfahren zur numerischen Lösung der Randwertaufgaben vom elliptischen Typus. Acad. Serbe Sci. Publ. Inst. Math. **9** (1956), 69–78.

The author seeks the solution of (*) $\phi_{xx}+\phi_{yy}+h\phi=-f$ in a closed bounded region S along whose boundary ϕ vanishes. He covers S with a rectangular mesh (lines and columns need not be equally spaced). Solving $\phi_{xx}=-p$ for each row, he expresses ϕ at every mesh-point in terms of p; similarly, solving $\phi_{yy}=-q$ for each column gives ϕ everywhere in terms of q; these two equations plus (*) in the form $p+q-h\phi=f$ yield three equations for ϕ, p, q. As an example, the author considers the torsion of an I-beam.

M. A. Hyman (Yorktown Heights, N.Y.).

Albrecht, J.; und Uhlmann, W. Differenzenverfahren für die 1. Randwertaufgabe mit krummlinigen Rändern bei $\Delta u(x, y)=r(x, y, u)$. Z. Angew. Math. Mech. **37** (1957), 212–224. (English, French and Russian summaries)

The authors develop formulas for the numerical solution of $u_{xx}+u_{yy}=r(x, y, u)$ in a region with a curved boundary along which u is specified. From Taylor series expansions they systematically obtain the coefficients (and order of approximation) for stars with 4, 6, 8 arms of different length. They emphasize the advantage of using more points in the star to reduce the truncation error without using too fine a mesh. The paper concludes with the solution of Laplace's equation over a semi-circle using various meshes and difference approximations.

M. A. Hyman (Yorktown Heights, N.Y.).

Durand, Émile. L'approximation du sixième ordre dans le calcul numérique des solutions de l'équation de Poisson à trois variables. C. R. Acad. Sci. Paris **245** (1957), 788–791.

The author derives some finite difference expressions in a cubic net approximating the partial differential equation $\partial^2u/\partial x^2+\partial^2u/\partial y^2+\partial^2u/\partial z^2=f(x, y, z)$. In particular, an expression is given involving all the mesh points on a cube of length $2h$, with center at the considered point, which is exact up to terms of order h^6.

U. Hochstrasser (Lawrence, Kans.).

Bellman, Richard. On the non-negativity of solutions of the heat equation. Boll. Un. Mat. Ital. (3) **12** (1957), 520–523.

The author remarks that the method of finite differences for the solution of the heat equation leads to a non-negative solution for non-negative boundary values.

M. Steinberg (Culver City, Calif.).

Ustinova, N. N. On a modification of the method of straight lines. Uč. Zap. Kazan. Univ. **115** (1955), no. 14, 159–167. (Russian)

The author considers the Dirichlet problem

$$\partial^2U/\partial x^2+\partial^2U/\partial y^2=0,\ U(p)=f(p) \text{ for } p\in\Gamma$$

in a region D bounded by a curve Γ which has only one point of intersection with every ray from the origin. Transforming to polar coordinates ρ, ϕ, making the transformation $t=\ln\rho$ and setting $U_k(t)=U(\rho,\phi_k)$, $\phi_k=2\pi k/n$, $(k=1, 2, \cdots, n)$ and then using the method of finite differences, we arrive at n approximative linear equations with constant coefficients for the functions $U_k(t)$. In the example considered here, the region D is a circle with center at the origin and $n=12$. The questions whether the method is suitable when D is not a circle and whether in that case the process converges as $n\to\infty$ are not considered.

Z. L. Leĭbenzon (RŽMat **1957**, no. 7390).

Ventcel', T. D. On the application of the method of finite differences to the solution of the first boundary problem for equations of parabolic type. Mat. Sb. N.S. **40(82)** (1956), 101–122. (Russian)

The author considers the quasi-linear parabolic equation

$$(1)\quad \frac{\partial^2u}{\partial x^2}=A(x,t,u)\frac{\partial u}{\partial t}+B(x,t,u)\frac{\partial u}{\partial x}+F(x,t,u)$$

in the rectangle R: $0\leq x\leq X$, $0\leq t\leq T$, subject to the conditions (2) $u(x,0)=u(0,t)=u(X,t)=0$. It is assumed that $A\geq a>0$ in R for $|u|\leq U$, and that f is bounded in R for all u. The following theorem is proved. If the coefficients of (1) are sufficiently regular, then there exists $T_0\leq T$ such that a unique regular solution of (1), (2) exists for $0\leq x\leq X$, $0\leq t\leq T_0$, and satisfies $|u|\leq U$. An analogous result holds if (1) is linear, with $T_0=T$ and non-zero boundary data. This result extends the work of E. Rothe [Math. Ann. **102** (1930), 650–670] on the semi-linear equation. The proof is carried out by considering the sequence of functions $v_{ij}^{(n)}=v^{(n)}(x_i,t_j)$, where (x_i,t_j) is a point in R on a grid of width $\Delta x_n=2^{-n}X$, $\Delta t_n=k(\Delta x_n)^2$, which satisfy the difference equation

$$(3)\quad \frac{\Delta^2v_{i-1,j}^{(n)}}{\Delta x^2}=A_{ij}\frac{\Delta v_{ij}^{(n)}}{\Delta x}+B_{ij}\frac{\Delta v_{ij}^{(n)}}{\Delta t}+F_{ij},$$

$$v_{i0}^{(n)}=v_{0j}^{(n)}=v_{2^n,j}^{(n)}=0,$$

where $A_{ij}=A(x_i,t_j,v_{ij}^{(n)})$, etc. and $\dfrac{\Delta}{\Delta x}$ is a forward difference in the x-direction. The author shows that if $1-2k/a>0$, then $v_{ij}^{(n)}$ and its first and second differences are uniformly bounded with respect to n, and concludes from this, by a standard argument, the existence of u as asserted. The uniqueness follows from the maximum principle for the (linear) equation of variation.

The author also considers the convergence of difference approximations to a known solution of (1), (2). In particular, if (1), (2) has a solution u in R such that $|u|\leq U$ and if the coefficients of (1) are bounded, continuous and Lipschitz continuous with respect to u in R for $|u|\leq U+c$ $(c>0)$, then $\lim_{n\to\infty}[u(x_i,t_j)-v_{ij}^{(n)}]=0$ uniformly, provided that $1-2k/a>0$. A similar result holds for the corresponding four-point implicit difference equation with no restriction on k.

D. G. Aronson.

Dreyfus, Stuart E. Computational aspects of dynamic programming. Operations Res. **5** (1957), 409–415.

The author discusses the functional equation technique of the theory of dynamic programming and some of the problems involved in using this technique to obtain computational results. Due to the limited capacity of digital computers, multi-dimensional processes cannot always be treated in a straightforward fashion. The author discusses in turn the technique of polynomial approximation, the method of Lagrange multipliers [cf. Bellman, Proc. Nat. Acad. Sci. U.S.A. **42** (1956), 767–769; MR **18**, 548], the application of search procedures, the study of analytic structure of the optimal policies, the use of dual processes, and finally the freedom to use either forward or backward equations; cf. T. F. Cartaino and S. Dreyfus, Application of Dynamic Programming to the Airplane Minimum Time-to-Climb Problem, Aeronautical Engineering, 1957.

R. Bellman.

Koval', I. Ĭ. On stability of solutions of various difference equations of mathematical physics. Kiïv. Derž. Ped. Inst. Nauk. Zap. **16**, Fiz.-Mat. Ser. no. **5** (1954), 3–11. (Ukrainian)

Adachi, Ryuzo. On the numerical solutions of some integro-differential equations under some conditions. Kumamoto J. Sci. Ser. A. **2** (1956), 322–335.

The author seeks numerically the solution $y(x)$ of the integro-differential equation

$$f+\int_a^b\varphi\,d\xi=0;$$

here f, φ are functions of x, y, y', y'' (primes denote derivatives). The interval $[a, b]$ is divided into m equal parts, the integral is replaced by a finite sum of φ's, and derivatives of y are expressed in terms of y at p nearby mesh-points. The resulting set of equations, when solved for the y's, yields the requisite solution. The author also gives error bounds. There are three carefully worked examples, two linear and one non-linear. The non-linear equations for y in the third example are solved iteratively. In each example, there is good agreement between the numerical solution and the (known) exact solution, well within the error bounds.

M. A. Hyman.

Belgrano Brémard, J. C. New nomographic methods. Las Ciencias **18** (1953), 5–22. (Spanish)

This paper is chiefly a summary of the work done by the author and his collaborators on nomographic methods, and provides a useful survey. The author refers to his work on tangential nomograms with a useful and simple technique to transform the nomogram. He describes the construction of the baricentric nomograms [López Nieto, Rev. Mat. Hisp.-Amer. (4) **11** (1951), 191–207; MR **13**, 592] and romboidal nomograms [Urcelay, Gac. Mat., Madrid **3** (1951), 183–194; MR **14**, 211] found by two of his collaborators. Finally, the author deals with the problem of identification of any relation, reducing it always to the identification of a relation given by tables,

and summarizes some of his work [ibid. 3 (1951), 143–152; MR 13, 389].

J. Vicente Gonçalves and J. Tiago de Oliveira.

Kreĭnes, M. A.; Vaĭnštein, I. A.; and Aĭzenštat, N. D. On nomographing functions given on a net. Dokl. Akad. Nauk SSSR (N.S.) **111** (1956), 941–944. (Russian)

Reutter, F. Nomographische Darstellung von Funktionen einer komplexen Veränderlichen. Z. Angew. Math. Mech. **36** (1956), 258–260.

Smirnov, S. V. On nomographability of equations. Ivanov. Gos. Ped. Inst. Uč. Zap. Fiz.-Mat. Nauki **4** (1953), 22–60. (Russian)

Vil'ner, I. A. Stereoscopic nomography and the solution of the problem of general anamorphosis in N-space. Uspehi Mat. Nauk (N.S.) **11** (1956), no. 4(70), 123–130. (Russian)

★ **Абрамов, А. А. [Abramov, A. A.] Таблицы $\ln \Gamma[z]$ в комплексной области. [Tables of $\ln \Gamma[z]$ in the complex plane.]** Izdat. Akad. Nauk SSSR, Moscow, 1953. 333 pp. 30.20 rubles.

This contains tables to six decimal places of the real and imaginary parts of $\ln \Gamma(z)$, $z=x+iy$, for $x=1.00(0.01)2.00$, $y=0.00(0.01)4.00$. To facilitate interpolation, values of Δ_2 (the smoothed second tabular difference with respect to y) are also given, together with a nomogram for calculating a correction term $\alpha(\xi, \eta)=\frac{1}{2}(\xi-\eta)(\xi+\eta-1)\Delta_2$. If $\beta(x, y)$ is the bilinear function which matches the table at the four points (x_0, y_0), (x_0+h, y_0), (x_0, y_0+h), (x_0+h, y_0+h), it is claimed that (in the rectangle of which these four points are the corners) $\beta(x, y)$ will be accurate to four decimal places, whereas $\beta(x, y)-\alpha(\xi, \eta)$, $\xi=(x-x_0)/h$, $\eta=(y-y_0)/h$ will be accurate to six decimal places. The introduction discusses methods of extending the table to values outside its range, and describes the method of computation. *H. B. Curry* (University Park, Pa.).

Ling, Chih-Bing. Tables of values of 16 integrals of algebraic-hyperbolic type. Math. Tables Aids Comput. **11** (1957), 160–166.

The integrals are $\int_0^\infty f(k, x)\{k!(\sinh 2x \pm \sin 2x)\}^{-1}dx$, where $f(k, x)$ has the following values for the integral values of k for which they are convergent: $(2x)^k$, $(2x)^k e^{-2x}$, $(2x)^k \tanh x$, $(2x)^k \coth x$, $x^k \sinh x$, $x^k \cosh x$, $x^k \tanh x \sinh x$, $x^k \coth x \cosh x$. The limiting value, as $k\to\infty$, for all except the second is unity; the asymptotic value of the second is $2^{-(k+1)}$. Values are given to 6D, and tabulation is continued until the limiting values are attained, which is the case when k is between 15 and 25. The two integrals for $f(k, x)=(2x)^k$, and the two for $f(k, x)=(2x)^k/2^{2x}$ were studied by R. C. J. Howland and tabulated by him [Philos. Trans. Roy. Soc. London. Ser. A. **229** (1930), 49–86] and by Howland and A. C. Stevenson [ibid. **232** (1933), 155–222]; further tabulations were made by the present author and C. W. Nelson [Ann. Acad. Sinica, Taiwan no. 2 (1955), part 2, 45–50]. All the others can be expressed in terms of these and the Glaisher Series $1\pm 2^{-k}+3^{-k}\pm 4^{-k}+\cdots$, $1\pm 3^{-k}+5^{-k}\pm 7^{k}+\cdots$. Various functional relations between the sixteen integrals were used for checking purposes.

These integrals occur in elastic problems concerning perforated or notched strips. *John Todd.*

★ **Yano, Tsuneta. Yano's tables of calculation.** The Tsuneta Yano Memorial Society, Tokyo, 1958. vi+162 pp. (2 plates) $2.50.

This collection of elementary tables was designed by the author to popularize the art of computation and they were calculated by him on abacuses after his retirement from the presidency of an insurance company. About half the book is devoted to detailed descriptions in English of the tables, with illustrative examples.

The tables include the following: $N=AB$ for $A=1(1)100$, $B=2(1)99$, 20 pages; $100N^{-1}$ for $N=1(1)1000$, 3D, 2 pages; A^2, A^3, $A^{\frac{1}{2}}$, $(10A)^{\frac{1}{2}}$, $A^{\frac{1}{3}}$, $(10A)^{\frac{1}{3}}$, $(100A)^{\frac{1}{3}}$ for $A=1(1)100$, 4D, 2 pages; $\sin\theta$, $\cos\theta$, $\tan\theta$ for $\theta=1(1)90°$, 5D, 1 page; factors of numbers between 1000 and 9999 not divisible by 2 or 5, 9 pages; a table for testing primarity, 2 pages; $\log N$, N prime $1000<N<10000$, 7D, with proportional differences, 8 pages; $\log N$, $N=100(1)999$, 15D, 2 pages; $\log N$, $N=1(1)150$, 7D, 1 page; $\log N$, $N=99900(1)99999$, 15D, with Δ, $-\Delta^2$, 2 pages; antilog x, $x=0(.001).999$, 4S, 2 pages; colog N, $N=1(1)999$, 4S, 2 pages; conversion table, NM and N/M for $N=1(1)100$, $M=\log_e 10$, 1 page.

There is a two page collection of formulae and a 22 page conversion table between the Japanese, the British-American, and the Metric systems.

This appears to be, essentially, an English version of the author's Kokumin sûhyô (People's tables) [Kokuseisha, Tokyo, 1952; MR **14**, 798; See also the review by D. H. Lehmer, Math. Tables Aids Comput. **7** (1953), 237–238]. *John Todd* (Pasadena, Calif.).

★ **Кармазина, Л. Н.; и Курочкина, Л. В. [Karmazina, L. N.; i Kuročkina, L. V.] Таблицы интерполяционных коэффициентов. [Tables of interpolation coefficients.]** Izdat. Akad. Nauk SSSR, Moscow, 1956. 365 pp. 7 plates. 37.65 rubles.

This book contains tables for the coefficients of the Lagrange, Newton, Bessel, Stirling and Everett formulas. In the introduction each formula with its remainder term is discussed and the relevant relations and tables of bounds for the errors are given. Some examples illustrate the use of the tables. The following coefficients are tabulated:

1) Coefficients for the Lagrange formula

$$L_k^n(t):\ t=\begin{cases}-\frac{1}{2}(n-2)\ (0.001)\frac{1}{2}n \text{ for } n \text{ even}\\ -\frac{1}{2}(n-1)\ (0.001)\ \frac{1}{2}(n-1) \text{ for } n \text{ odd}\end{cases} \qquad n=3(1)8,\ 10\text{D}$$

and a short table of the same for

$$t=\begin{cases}-\frac{1}{2}(n-2)\ (0.1)\frac{1}{2}n \text{ for } n \text{ even}\\ -\frac{1}{2}(n-1)\ (0.1)\frac{1}{2}(n-1) \text{ for } n \text{ odd}\end{cases} \qquad n=3(1)11,\ 3\text{–}10\text{D}$$

2) Coefficients for the Newton formula

$$N_k=\frac{t(t-1)\cdots(t-k+1)}{k!};$$

$t=0(0.001)1$; $k=2(1)7$; 10D

3) Coefficients for the Bessel formula

$$B_{2k}=\frac{t(t^2-1)\cdots[t^2-(k-1)^2](t-k)}{(2k)!}$$

$$B_{2k+1}=\frac{t(t^2-1)\cdots(t^2-(k-1)^2)(t-k)(t-\frac{1}{2})}{(2k+1)!}$$

$t=0(0.0001)1$; $k=1, 2, 3$; 10D

4) Coefficients for the Stirling formula:

$$S_{2k}=\frac{t^2(t^2-1)\cdots(t^2-(k-1)^2)}{(2k)!}$$

$$S_{2k+1}=\frac{t(t^2-1)\cdots(t^2-k^2)}{(2k+1)!}$$

$t=0(0.001)1$; $k=1, 2, 3$; 10D

5) Coefficients for the Everett formula

$$E_{2k}=\frac{t(t^2-1)\cdots(t^2-k^2)}{(2k+1)!}$$

$t=0(0.001)1$; $k=1, 2, 3$; 10D

At the end nomograms for quadratic and cubic interpolation with the Newton, Bessel and Stirling formulas are given, which are convenient for hand computations. The tables were computed anew, exept for a part of the tables for the Lagrangian coefficients, which was copied from "Tables of Lagrangian interpolation coefficients" [Nat. Bur. Standards, Columbia Univ. Press, 1944; MR 5, 244]. *U. W. Hochstrasser* (Lawrence, Kans.).

★**Канторович, Л. В.; Крылов, В. И.; и Чернин, К. Е. [Kantorovič, L. V.; Krylov, V. I.; i Černin, K. E.] Таблицы для численного решения граничных задач теории гармонических функций. [Tables for numerical solution of boundary problems in the theory of harmonic functions.]** Gosudarstv. Izdat. Tehn.-Teor. Lit., Moscow, 1956. 462 pp. 18.85 rubles.

For plane regions bounded by sufficiently regular contours Γ, the first boundary-value problem (Dirichlet problem) with boundary function f is solved by the formula

$$u(z)=\int_\Gamma \frac{\partial G(s,z)}{\partial n} f(s)ds.$$

Replacing $f(s)$ by a suitable interpolation polynomial yields

$$u(z)\approx\sum_{i=1,3,\cdots}^{n-1}\{A_i^{(-1)}(z)f(s_{i-1})+A_i^{(0)}(z)f(s_i)+A_i^{(1)}(z)f(s_{i+1})\}.$$

An approximate solution of the Dirichlet problem for a fixed region with arbitrarily prescribed boundary values can thus be obtained from tabulated values of the coefficient functions $A_i^{(-1)}$, $A_i^{(0)}$, $A_i^{(1)}$. The present work provides such tables for the following regions: (1) rectangle, (2) half-plane, (3) strip, (4) halfstrip, (5) angle, (6) circle, (7) half-circle. Similar tables are included for the normal derivative and for the second boundary-value problem (Neumann problem) (the latter only for the case of a rectangle). Unfortunately, the tables contain no headings, other than the numerical entries. Moreover, while the 142 pages of text preceding the tables discuss the theory, aided by numerous illustrative examples, the work suffers from lack of a clearly presented summary of formulas and instructions for using the tables.

M. G. Arsove (Seattle, Wash.).

★**Вазорова. А. И. [Vzorova, A. I.] Таблицы для решения уравнения Лапласа в эллиптических областях. [Tables for solution of the Laplace equation in elliptic domains.]** Izdat. Akad. Nauk SSSR, Moscow, 1957. 257 pp. 30 rubles.

The tables compiled here are similar to those of the preceding review but apply only to the Dirichlet problem for an ellipse. However, in this case the tabulated values are clearly designated by appropriate headings and a concise explanation of the use of the tables is given.

M. G. Arsove (Seattle, Wash.).

Huss, Carl R.; and Donegan, James J. Tables for the numerical determination of the Fourier transform of a function of time and the inverse Fourier transform of a function of frequency, with some applications to operational calculus methods. NACA Tech. Note no. 4073 (1957), 205 pp.

If the function $f(t)$ is zero for negative values of t and if, for positive values it is fitted with a "staircase" function by using equal intervals of time Δt, then its Fourier transform is $\Delta t \sum_{n=1}^{\infty} r_n\phi_n(z)$, where r_n is the amplitude of the nth step of the staircase function, and $\phi_n(z)=z^{-1}\sin z\exp[(2n-1)iz]$, with $z=\frac{1}{2}\omega\Delta t$. In this publication the real and imaginary parts of $\phi_n(z)$ are tabulated for $n=1$ to 100 and $z=0.00(0.01)5.08$. Usually 100 intervals are sufficient but, if they are not, more can be used by direct use of the tables and the shift theorem; this process is described in the explanatory text accompanying the tables. The tables are easily adapted to finding the inverse Fourier transform of a function of frequency.

In the accompanying text the authors point out how many of the operations performed by operational methods can be cast into purely numerical form and illustrate their remarks by describing briefly some practical applications.

I. N. Sneddon (Glasgow).

See also: Linear Algebra: Ishaq; Lotkin. **Polynomials:** Kulik. **Special Functions:** Kreyszig. **Partial Differential Equations:** Ryaben'kiĭ and Filippov. **Computing Machines:** Oblonský. **Mechanics of Particles and Systems:** Apetaur and Půst. **Elasticity, Plasticity:** Béres; Béres, Lovass-Nagy and Szabó; Langefors; Bishop and Johnson. **Programming, Resource Allocation, Games:** Prager; Bellman.

Computing Machines

Lunelli, Lorenzo. Un comando di computo di cicli per una calcolatrice elettronica aritmetica a tre indirizzi. Ricerca Sci. **27** (1957), 3381–3394.

Si descrive un nuovo comando per il computo dei cicli aggiunto alla calcolatrice elettronica aritmetica CRC 102A in dotazione al Centro di Calcoli Numerici del Politecnico di Milano. Illustrati i circuiti per la realizzazione se ne mostrano alcune applicazioni come contatore e come modificatore di indirizzi. *Riassunto dell'autore.*

Klika, Otakar. Common problems of telecommunications and mathematical machines. Stroje na Zpracování Informací **3** (1955), 15–30 (1956). (Czech. Russian and English summaries)

Digital computers and telephone switchboard networks are compared. They have many resemblances and common functional parts, such as memory units, switching equipment for setting up a connected path, etc. In switchboards, the numerical information supplied by dialing is often translated into another number system. The use of vacuum tubes, transistors and relays is discussed briefly. *V. Vand* (University Park, Pa.).

Černý, Václav; and Oblonský, Jan. Machine for computation of crystal structures. Stroje na Zpracování Informací **3** (1955), 31–47 (1956). (Czech. Russian and English summaries)

A special-purpose digital computer with built-ni

instructions has been constructed for the computation of crystal structure factors. The three-dimensional atomic co-ordinates are read in through a keyboard. The Miller indices and the atomic scattering factors are read in by Powers punched cards. The squares of the moduli of the calculated and observed structure factors (the latter being punched on the same cards as the Miller indices) are compared and a disagreement function

$$W=\sum_{i=1}^{M-1}|K_{i+1}-K_i|$$

is computed, where M is the total number of reflections and the ratios $K_i=F_e^2/F_0^2$ refer to the ith reflection with Miller indices h_i, k_i, l_i. The input coordinates are varied by trial and error until a sufficiently low value of W is obtained.

The machine uses about 1100 relays and works with binary numbers, at the rate of 40 arithmetic operations per second. The results can be printed on the attached printer. A flowdiagram and diagrams of some of the relay circuits used for multiplication and sine-cosine generation are given. The machine works in the space-group $P1$ and has capacity for 60 atoms.

V. Vand.

Oblonský, Jan. Machine for Fourier synthesis. Stroje na Zpracování Informací 3 (1955), 49–59 (1956). (Czech. Russian and English summaries)

A machine for crystallographic Fourier summation has been built, using a card sorter, an adding unit and a printer. It operates on the principle of Beevers-Lipson strips; a library of 74,400 punched cards is prepared, each card carrying sines or cosines for one Miller index h (range from 1 to 15), multiplied by the amplitude A or B (range in binary from -1023 to $+1023$), for 30 values of x (range 0 to 119/120 of the cell edge). The punched cards are of the Powers type. The numbers are in binary. Each card is reproduced ten times. The sorter is used as a binary card reader and not for sorting. The brush pulses are added progressively in the adder. Special cards cause the sotal to be printed and counter cleared. In this way, the tums

$$A_h\cos 2hx_l \text{ or } B_h\sin 2hx_l$$

are accumulated for the values of x_l given by the position of the brush of the sorter. Calculation of one Fourier map with a 120×120 grid takes about 34 machine hours. Hand picking of the cards takes about 22.5 hours. Cards begin to wear out after about 200 passages through the machine.

V. Vand (University Park, Pa.).

Svoboda, Antonín. Application of the Korobov sequence in mathematical machines. Stroje na Zpracování Informací 3 (1955), 61–76 (1956). (Czech. Russian and English summaries)

Theory of the Koborov [Izv. Akad. Nauk SSSR. Ser. Mat. 14 (1950), 215–238; MR 12, 321] sequence is discussed, and it is shown that it can be used for determination of addresses in the dynamic memory of digital computers, such as magnetic drums. A generalisation is made by defining an "extended Korobov sequence". A procedure is given for its construction and its existence is proved. It is then applied to the determination of the addresses on the magnetic drum memory of the Czechoslovak digital computer SAPO.

V. Vand.

★ **Wilkes, Maurice V.; Wheeler, David J.; and Gill, Stanley. The preparation of programs for an electronic digital computer.** 2nd ed. Addison-Wesley Mathematics Series. Addison-Wesley Publishing Company, Inc., Reading, Mass., 1957. xiv+238 pp. $7.50.

The first edition of this book [1951; MR 13, 162] was a significant contribution to the literature. The second edition, while broadened to cover other machines than the EDSAC, no longer brings the reader to the forefront of the field; in particular almost all the recent developments in automatic coding (which dominates the actual use of machines in America) are either glossed over or totally ignored in the book. From being outstanding when first issued it has fallen to being average in the second edition.

R. W. Hamming (Murray Hill, N.J.).

Trahtenbrot, B. A. On operators realizable in logical nets. Dokl. Akad. Nauk SSSR (N.S.) 112 (1957), 1005–1007. (Russian)

Let X, Z, Q be three finite alphabets with m, n, k letters respectively. Let $x(t)$, $z(t)$, $q(t)$, $t=1, 2, 3, \cdots$ be sequences (finite or infinite) of letters from X, Y, Q respectively. The paper concerns operators which transform an input sequence $x(t)$ into an output sequence $z(t)$ using $q(t-1)$ as a memory storage for one unit of t. The transformation thus is of the form

$$z(t)=F(x(t), q(t-1)),\ q(t)=G(x(t), q(t-1)),$$

where F and G are functions with appropriate ranges defined for x in X, q in Q, and $q(0)$ is an arbitrary constant. The minimal value of k (supposed finite) is called the weight of the operator. Then $\log_2 k$ represents the measure of information given by a character of Q, and $\mu\log_2 m$ that of an input word of length μ; the ratio of these two is called the specific weight of the operator. The author states (without proof) that for every $\varepsilon>0$, the proportion of cases where the specific weight $<1-\varepsilon$ approaches 0 as $\mu\to\infty$; that an operator with weight k transforms a periodic input with an initial word of length p followed by period (i.e. repetitions of a word of length) r into one with initial length p' and period r', where $r'\leq kr$ and $p'+r'\leq p+kr$; and that a necessary and sufficient condition that two operators of weight $\leq k$ coincide is that they have the same effect on all input words of length $\leq 2k-1$.

H. B. Curry (University Park, Pa.).

Luhn, H. P. The automatic creation of literature abstracts. IBM J. Res. Develop. 2 (1958), 159–165.

In the system proposed in this paper the article to be abstracted is prepared in machine-readable form and scanned by a computer, which first compiles a frequency list of the words it finds. Words with frequencies less than or greater than preassigned limits are ignored, and the remaining words are used to measure "significance". Sentences which have runs of frequently occurring words are considered significant, and three or four of these sentences are quoted to form the abstract. Two examples are given of the original articles and the abstract formed from them.

C. C. Gotlieb (Toronto, Ont.).

★ **Герлах, Л. Р. [Gerlah, L. R.] Некоторые вопросы проектирования механичееких интегро-дифференцнографов. [Some questions on the planning of mechanical integro-differentiators.]** Gosudarstv. Izdat. Oboronn. Promyšl., Moscow, 1957. 53 pp. 2 rubles.

Raymond, François-Henri. Les analyseurs differentiels electroniques. Calc. Automat. y Cibernet. **5** (1956), no. 13, 1–31.

This is a reprint of an introductory, summarising paper given by the author at the opening of the International Conference on Analogue Computing (Brussels, 1955) and published in the Proceedings of that Conference (Brussels, 1956). It is marred by so many errors and omissions (including a few from the original) that the reader is advised to have recourse to the original publication.

As a paper reviewing in great generality the principles of analogue calculation, its coverage is broad rather than detailed. The philosophy of analogue computation, and its main advantages are considered; mechanical, electronic and digital differential analysers are discussed. It presents the principles of setting up an analogue computer from a topological viewpoint — mainly with reference to electronic differential analysers, which have been the author's special field. *J. G. L. Michel* (Teddington).

See also: General Theory of Numbers: Leech; Dijkstra. **Numerical Methods:** Valat; Gillies and Hunt. **Statistics:** Shiskin and Eisenpress. **Structure of Matter:** Vlasenko and Zhdanov; Cochran and Douglas.

PROBABILITY

Lévy, Paul. Une nouvelle classe de fonctions symboliques: les σ-fonctions. Bull. Sci. Math. (2) **80** (1956), 83–96.

Intuitivement, une σ-fonction est un être mathématique de la forme $\theta=f\sqrt{\mu}$, μ mesure de Radon $\geqq 0$ sur un espace localement compact X dénombrable à l'infini, $f\in L^2(\mu)$; alors $|\theta|^2$ est la mesure $|f|^2\mu\geqq 0$, et plus généralement le produit de deux σ-fonctions est une mesure; on peut définir la somme de deux σ-fonctions; la norme $\|\theta\|=\int|f|^2d\mu$ fait de l'espace des σ-fonctions sur X un espace de Hilbert. Pour donner une définition rigoureuse, on considèrera l'ensemble des couples (f,μ), ν mesure $\geqq 0$, $f\in L^2(\nu)$, et la relation d'équivalente $(f,\alpha\mu)\sim(g,\beta\mu)$, si α et β sont des fonctions $\geqq 0$ localement μ-intégrables et $f\sqrt{\alpha}=g\sqrt{\beta}$ μ-presque partout; une σ-fonction est une classe d'équivalence, la classe de (f,ν) pouvant se noter $f\sqrt{\nu}$. L'auteur étudie les principales propriétés des σ-fonctions et donne des applications à la représentation des fonctions aléatoires. Le reviewer donne une autre présentation de la théorie de l'auteur, dans un article à paraître prochainement (en hommage à l'auteur). *L. Schwartz.*

Pèrez, Albert. Transformation ou σ-algèbre suffisante et minimum de la probabilité d'erreur. Czechoslovak Math. J. **7(82)** (1957), 115–123. (Russian summary)

Let $M=\{\mu_i, i=1,\cdots,n\}$ be probability measures on a measurable space (X,S) with corresponding a priori probabilities $P=\{p_i\}$, $\sum p_i=1$. The probability of error in deciding that the true measures are μ_i when $x\in B_i$, where the B_i form a (measurable) partition of X, is defined by $1-\sum p_i\mu_i(B_i)$. A family of partitions is obtained with minimal probability e of error. Then a nondecreasing sequence S_k of sub-σ-algebras and corresponding minimal probabilities e_k of error are considered. It is proved that $e_n\to e$ if the σ-algebra generated by $\bigcup S_k$ is sufficient with respect to M, in fact, that, whatever be P, the minimal probability of error relative to a sub-σ-algebra S_0 coincides with e if and only if S_0 is sufficient with respect to M. *M. Loève* (Berkeley, Calif.).

Rosenblatt, David. On the graphs and asymptotic forms of finite Boolean relation matrices and stochastic matrices. Naval. Res. Logist. Quart. **4** (1957), 151–167.

Finite dimensional Boolean relation matrices are studied, i.e., square matrices of ones and zeros, multiplied by using sup and inf instead of ordinary addition and multiplication. Asymptotic forms of the powers of such matrices are obtained in terms of "cyclic nets" of indices, i.e., sets S of indices such that if $i,j\in S$ then there exist $i_1,\cdots,i_k$ such that the matrix elements $a_{ii_1}, a_{i_1i_2},\cdots,a_{i_kj}$ are all 1. Given a stochastic matric P, one gets a relation matrix $G(P)$ by putting a zero in $G(P)$ in exactly those places where P has a zero.Some of the asymptotic properties of P^n can be obtained from asymptotic properties of $G(P)^n$, since $G(P^n)=G(P)^n$, and such a study is carried out. *J. Feldman* (Berkeley, Calif.).

Ludwig, Otto. Die Pascalsche Fragestellung für Merkmalsiterationen. Mitteilungsbl. Math. Statist. **9** (1957), 1–26, 81–101.

Given a sequence of observations from a binomial or multinomial population. The paper considers questions of the type: what is the probability that the αth run (α fixed) of length exactly m or $\geqq m$ of elements of a specific kind or of all kinds occurs at the kth trial? Probabilities are given either explicitly or in terms of generating functions. Cumulants up to the 4th order are given, as well as certain asymptotic approximations. *G. E. Noether.*

★ **Гнеденко, Б. В.; и Хинчин, А. Я. [Gnedenko, B. V.; i Hinčin, A. Ya.] Элементарное введение в теорию вероятностей. [Elementary introduction to the theory of probability.]** 4th ed. Gosudarstv. Izdat. Tehn.-Teor. Lit., Moscow, 1957. 144 pp. 2.20 rubles.

The 3rd edition is listed in MR **14**, 293. The 4th is practically unchanged.

Linnik, Yu. V. On the composition of Gaussian and Poissonian probability laws. Dokl. Akad. Nauk SSSR (N.S.) **114** (1957), 21–24. (Russian)

The author sketches a proof of the following theorem. If the composition (convolution) of a Gaussian and a Poisson distribution is expressed as a composition of two distributions, each of the latter distributions is itself the composition of a Gaussian and a Poisson distribution. Moreover, the variances of the Gaussian [Poisson] components have as sum the variance of the original Gaussian [Poisson] component. If the original distribution is Gaussian, the theorem reduces to the Cramér-Lévy theorem [Cramér, Math. Z. **41** (1936), 405–414] that two distributions must be Gaussian if their composition is. If the original distribution is Poisson, the theorem reduces to the corresponding one of Raikov for Poisson distributions [Izv. Akad. Nauk SSSR. Ser. Mat. **1938**, 91–124].

J. L. Doob (Urbana, Ill.).

Lukacs, Eugene. Remarks concerning characteristic functions. Ann. Math. Statist. **28** (1957), 717–723.

The author shows that the weighted sum of a sequence of characteristic functions is itself a characteristic func-

tion if and only if the weights are non-negative and sum to unity. He then derives a new class of characteristic functions for infinitely divisible distributions, and rederives a class due to de Finetti [Atti Accad. Naz. Lincei. Rend. Cl. Sci. Fis. Mat. Nat. (6) **12** (1930), 278–282]. Having proved in an earlier paper [Pacific J. Math. **2** (1952), 615–625; MR **14**, 485] that an analytic characteristic function of an infinitely divisible distribution has no zeros inside its strip of convergence, he shows that such characteristic functions may have zeros arbitrarily near, and indeed on, the boundary of the strip. The paper concludes with an example of a not infinitely divisible distribution whose characteristic function has nonetheless a non-enumerable set of factors, themselves characteristic functions, depending on a continuous parameter. This answers affirmatively a question raised by Dugué [Ann. Inst. H. Poincaré **12** (1951), 159–169; MR **13**, 759].

S. Katz (New York, N.Y.).

Spitzer, Frank. On interval recurrent sums of independent random variables. Proc. Amer. Math. Soc. **7** (1956), 164–171.

Let $\{S_n\}$ be the sequence of partial sums of mutually independent, identically distributed random variables. Under essentially the conditions assumed by Kallianpur and Robbins [Duke Math. J. **21** (1954), 285–307; MR **16**, 52] the following results are proved:

$$(1)\quad \lim_{n\to\infty} \Pr\{\min_{1\le k\le n} |S_k-a|\le xC(n,\alpha)\} = \begin{cases} 0 \text{ for } x\le 0, \\ 1-E_{1-1/\alpha}(-x) \text{ for } x\ge 0, \end{cases}$$

where $B_p(z)$ is the Mittag-Leffler function $\sum_{n=0}^{\infty} z^n/(\Gamma(1+pn))$, and

$$C(n,\alpha) = \begin{cases} \tfrac{1}{2}\alpha \sin\dfrac{\pi}{\alpha}\, n^{1/\alpha-1} & \text{for } 1<\alpha\le 2, \\ \tfrac{1}{2}\pi(\log n)^{-1} & \text{for } \alpha=1; \end{cases}$$

$$(2)\quad \Pr\{|S_n-a|\le n^{1/\alpha}a_n \text{ for infinitely many } n\}=0 \text{ or } 1$$

for $\sum_1^{\infty} a_n$ convergent or divergent respectively, $\{a_n\}$ being a nonincreasing sequence of positive numbers. The proofs of both theorems use an estimate, for large n, of the density $f_n(x)$ of S_n which was obtained by the authors cited above and employed by them to prove the interval recurrence and equidistribution of $\{S_n\}$. *G. Kallianpur.*

Chatterjee, S. D.; and Pakshirajan, R. P. On the unboundedness of infinitely divisible laws. Sankhyā **17** (1957), 349–350.

The authors show that a bounded distribution cannot be infinitely divisible.

H. B. Mann.

Spitzer, Frank. The Wiener-Hopf equation whose kernel is a probability density. Duke Math. J. **24** (1957), 327–343.

The author considers (*) $F(x)=\int_0^{\infty} k(x-y)dF(y)$ where k is a known probability density function, and the more general equation (**) $F(x)=\int_{-\infty}^{x} F(x-u)dG(u)$ where G is a known d.f. A solution F is said to be a P^* solution if it is right continuous, nondecreasing, not identically zero, and zero for $x<0$; if also F is a d.f., it is said to be a P-solution. To avoid trivialities, assume G does not have a jump of 1 at 0. Lindley [Proc. Cambridge Philos. Soc. **48** (1952), 277–289; MR **13**, 759] showed that, if G has a finite first moment μ, then (**) has a unique P-solution or no P-solution according to whether (1) $\mu<0$ or $\mu\ge 0$. Let S_k be the sum of k independent random variables with common d.f. G. With no assumption about moments, the author replaces (1) by the condition (2) $\sum_k P\{S_k>0\}/k<\infty$ or $=\infty$. A method of obtaining the Laplace transform of F is given. It is shown that (*) has a unique P^*-solution (to within a multiplicative constant) if $\sum P\{S_k\le 0\}/k=\infty$ or if $\mu\le 0$. If k is even, then (*) has a unique nondecreasing solution with $F(0+)=1$, whose Laplace transform is found; if k has finite (infinite) variance, then, as $x\to\infty$, $F(x)\sim 2^k x/\sigma$ (resp., $o(x)$).

J. Kiefer (Ithaca, N.Y.).

Chernoff, H.; and Daly, J. F. The distribution of shadows. J. Math. Mech. **6** (1957), 567–584.

Suppose points are randomly scattered (Poisson process) in the plane, circles of radius r are located with those points as centers, and the shadow thrown by these circles from a point P onto a line L is considered. Let a point 0 and direction on L be fixed. The authors determine the distribution of the maximum uninterupted sunny or shady interval which can be traversed on L, starting from 0. The method is extended to consider nonconstant r, nonconstant density of centers, etc. Applications are given to certain traffic problems, circular counter problems, and type II counter problems. For example, results of Takács [Proc. Cambridge Philos. Soc. **52** (1956), 488–498; MR **18**, 424] on the latter are obtained. *J. Kiefer.*

Kozin, Frank. A limit theorem for processes with stationary independent increments. Proc. Amer. Math. Soc. **8** (1957), 960–963.

Let $[X(t); 0\le t\le 1]$ be a process with stationary independent increments having characteristic function $\exp[t\,h(u)]$. Using Baxter's procedure [Proc. Amer. Math. Soc., **7** (1956), 522–527], the author obtains the following theorem: Let $E[X(t)^4]$ exist and $D^4h(u)|_{u=0}=0$. If $0=t_0<t_1<\cdots<t_{N_n}=1$ and $\lim_{n\to\infty}\Delta t_{\max}=0(n^{-2})$, where $\Delta t_{\max}=\max(t_k-t_{k-1})$ for $k=1,\ \cdots,\ N_n$, then with probability one

$$\lim_{n\to\infty}\sum_{k=1}^{N_n}[X(t_k)-X(t_{k-1})]^2=-D^2h(0).$$

This result overlaps Baxter's theorem in the case of Gaussian processes with stationary independent increments, i.e., Wiener processes. The author also notes that Baxter's theorem can be proved for more general t_k using his method. *H. P. Edmundson.*

Baxter, Glen. A strong limit theorem for Gaussian processes. Proc. Amer. Math. Soc. **7** (1956), 522–527.

Let $\{W(t), 0\le t\le 1\}$ be the Wiener process. It is a well known result due to Levy that with probability one

$$\lim_{n\to\infty}\sum_{k=1}^{2^n}\left[W\left(\frac{k}{2^n}\right)-W\left(\frac{k-1}{2^n}\right)\right]^2=1.$$

Let $\{x(t), 0\le t\le 1\}$ be a Gaussian process with mean function $m(t)$ and covariance function $r(s,t)$. Assume $m(t)$ has a bounded first derivative for $0\le t\le 1$ and that $r(s,t)$, is continuous in $0<s,\ t<1$ and has uniformly bounded second derivatives for $s\ne t$. Let

$$D^+(t)=\lim_{s\to t^+}\frac{r(t,t)-r(s,t)}{t-s},\quad D^-(t)=\lim_{s\to t^-}\frac{r(t,t)-r(s,t)}{t-s},$$

and $f(t)=D^-(t)-D^+(t)$. The main result of the paper is that with probability one

$$\lim_{n\to\infty}\sum_{k=1}^{2^n}\left[x\left(\frac{k}{2^n}\right)-x\left(\frac{k-1}{2^n}\right)\right]^2=\int_0^1 f(t)dt.$$

This includes the earlier result cited above. The special case of a factorable covariance function is studied, and in this connection the author proves that if $p(s)>0$, $q(s)\geq 0$, $p'(s)$ is continuous on $0\leq s\leq 1$, and h and H are extended real-valued numbers, and if the system:

$$\frac{d}{ds}\left\{p(s)\frac{dy}{ds}\right\}-q(s)y=0$$
$$y(0)-hy^1(0)=0$$
$$y(1)+Hy^1(1)=0$$

is incompatible, then the Green's function $r(s, t)$ is a covariance function of a Gaussian process.

M. D. Donsker (Minneapolis, Minn.).

Ueno, Tadashi. Some limit theorems for temporally discrete Markov processes. J. Fac. Sci. Univ. Tokyo. Sect. I. 7 (1957), 449–462.

Let $P(x, dy)$ be a Markov transition probability defined on (Ω, B), $P_{(x,y)}(dz)$ denote the set function $P(x, dz)-P(y, dz)$, and $\|P_{(x,y)}(dz)\|$ denote the total variation of $P_{(x,y)}(dz)$ (as a completely additive set function) on (Ω, B). The author defines $Q(P)=\frac{1}{2}\sup_{(x,y)}\|P_{(x,y)}(dz)\|$ and shows $Q(P\cdot P')\leq Q(P)Q(P')$, where $P\cdot P'(x, E)=\int_\Omega P'(y, E)P(x, dy)$. Using this inequality and making some assumptions about $Q(P)$, he proves theorems on almost sure convergence, zero-one laws, etc., for discrete time Markov processes. A discrete time Markov process is called of "mixing type with respect to $Q(P)$" if, for every integer m, $\lim_{n\to\infty} Q(P_m\cdot P_{m+1}\cdot\cdots\cdot P_{m+n})=0$. This definition is shown to agree with the usual definition of "mixing" type for stationary Markov processes, if one assumes Doeblin's condition (otherwise, the relationship between the two definitions is not clear).

M. D. Donsker (Minneapolis, Minn.).

Udagawa, Masatomo. On some limit theorems for the sums of identically distributed independent random variables. Kōdai Math. Sem. Rep. 8 (1956), 85–92.

Let $X_1, X_2, X_3, \cdots$ be identically lattice distributed independent random variables with $S_n=X_1+X_2+\cdots+X_n$, and N_n denoting the number of S_k, $1\leq k\leq n$, which are zero.

Assuming that $S_n/n^{1/\alpha}$ $(1<\alpha<2)$ converges in distribution to the symmetric stable distribution with exponent α, the author finds the limiting distribution of the N_n (suitably normalized). In this connection, and in particular for the method of proof, see Kallianpur and Robbins [Duke Math. J. 21 (1954), 285–307; MR 16, 52]. It is also shown that the frequency function form of the central limit theorem due to W. L. Smith [Proc. Cambridge Philos. Soc. 49 (1953), 462–472; MR 14, 1099] is true under the assumptions used by Gnedenko and Kolmogorov in the frequency form of the central limit theorem proved in their well-known book.

M. D. Donsker (Minneapolis, Minn.).

Billingsley, Patrick. The invariance principle for dependent random variables. Trans. Amer. Math. Soc. 83 (1956), 250–268.

In this paper the Erdös-Kac invariance principle as generalized by the reviewer [Mem. Amer. Math. Soc. no. 6 (1951); MR 12, 723] is extended to the dependent case. Let C be the space of functions continuous on the closed unit interval. Let $\rho(x, y)=\sup_{0\leq t\leq 1}|x(t)-y(t)|$, C be the Borel field generated by the open sets defined by ρ, and W be Wiener measure on C. Let $X_1, X_2, \cdots$ be a sequence of random variables on some probability space $(\Omega B, P)$. Let $S_n=X_1+X_2+\cdots+X_n$ and p_n be that element of C which is linear on each of the intervals $((j-1)n^{-1}, jn^{-1})$, $j=1, 2, \cdots, n$, and satisfies $p_n(jn^{-1})=S_j$, $j=0, 1, \cdots, n$. Thus p_n is a mapping of Ω into C and this mapping is measurable. Suppose there exists a sequence $\{a_n\}$ of positive constants such that if P_n is a measure defined by $P_n(A)=P\{a_n^{-1}p_n\in A\}$, for $A\in C$, then P_n converges weakly to W. Under these circumstances we say the invariance principle holds for the sequence $\{X_n\}$ with norming factors $\{a_n\}$. Donsker's earlier result showed the invariance principle holds with norming factors $n^{\frac{1}{2}}$ provided $\{X_n\}$ is an independent, stationary sequence with $E\{X_n\}=0$ and $E\{X_n^2\}=1$. The author replaces the independence assumption by weaker hypotheses, obtaining interesting new versions of the invariance principle. In particular he shows the invariance principle holds for discrete Markov processes satisfying Doeblin's condition, for m-dependent random variables, for discrete linear processes with m-dependent residuals and lastly if X_n is 0 or 1 according as a recurrent event does or does not occur at the nth of a sequence of trials. Each of these generalizations depends on the fact that in each case a central limit theorem can be proved under appropriate hypotheses.

M. Donsker (Minneapolis, Minn.).

Prékopa, András. On the convergence of series of independent random variables. Publ. Math. Debrecen 4 (1956), 410–417.

Let $X_1, X_2, X_3, \cdots$ be a sequence of independent random variables. Let K denote the class of all finite subsets of the positive integers and S denote the class of all subsets of the positive integers. Let $X(A)=\sum_{k\in A}X_k$ provided the series converges with probability one regardless of the order of summation. Let $F(x, A)$ denote the distribution function of the random variable $X(A)$. Let $\mathfrak{F}$ be the space of one-dimensional distribution functions with the Lévy metric. The author shows that if the subset of $\mathfrak{F}_1\{F(x, A), A\in K\}$ is compact, then $\sum_{k=1}^{\infty}X_k$ converges with probability one regardless of the order of summation. Also if $\sum_{k=1}^{\infty}X_k$ converges with probability one regardless of the order of summation, then the subset of $\mathfrak{F}_1\{F(x, A), A\in S\}$ is compact.

M. D. Donsker (Minneapolis, Minn.).

Theodorescu, Radu. Stochastische kontinuierliche Prozesse mit vollkommenen Verbindungen. Math. Nachr. 16 (1957), 79–84.

O. Onicescu found a differential equation for probabilities to hold for chains of complete connection [Rev. Univ. "C. I. Parhon" Politehn. Bucuresti. Ser. Şti. Nat. 3 (1954), no. 4–5, 73–85; MR 17, 980] under the assumption of a finite state space. Without this assumption, the author proves the existence and uniqueness of a solution of a corresponding equation by the method of successive approximations. This is said to extend a result in the quoted paper. Since this paper was not accessible to the reviewer, the exact relationship of both results is unknown to him.

H. M. Schaerf (Madison, Wis.).

Aczél, J.; et Egerváry, E. Remarques algébriques sur la solution donnée par M. Fréchet à l'équation de Kolmogoroff. II. Publ. Math. Debrecen 5 (1957), 60–71.

Continuing an earlier paper by Aczél [same Publ. 4 (1955), 33–42; MR 16, 989], the authors study the functional equation (1) $P(s, t)P(t, u)=P(s, u)$, valid for all real s, t, u, for finite-dimensional matrices. The solutions

must have the form $\Pi(t)^{-1}F\Pi(u)$, where $\Pi(t)$ is non-singular, and F is a matrix whose elements vanish except for ones in some of the places on the main diagonal. The added condition that $P(s, t)$ have row sums 1 corresponds to a simple condition on $\Pi(t)$. If, as in the case of probability matrices, $P(s, t)$ is defined only for $s \leqq t$, and if (1) is to hold only for $s \leqq t \leqq u$, the preceding results remain true. The interpretation in this normal form of the added probability condition that $P(s, t)$ have non-negative elements is not discussed. *J. L. Doob* (Urbana, Ill.).

Feller, William. Boundaries induced by non-negative matrices. Trans. Amer. Math. Soc. **83** (1956), 19–54.

Let E be the set of positive integers, and let Π be a matrix with non-negative elements and row sums $\leqq 1$. The author immerses E in a Hausdorff space, depending on Π, to obtain a boundary for E. A second boundary is also obtained, adjoint to the first and in no relationship to the first except by way of Π. The boundaries are to be applied in a subsequent paper to the theory of the Kolmogorov differential equations for the transition probabilities of a continuous parameter Markov process with countably many states. Let $\mathfrak{M}$ be the class of vectors z with $z \geqq 0$ and $\Pi z = z$. Let $\mathfrak{P}$ be the subclass of $\mathfrak{M}$ with $z \leqq 1$. The class $\mathfrak{P}$ is a convex set, and a lattice. The extremals of this convex set, called sojourn solutions, form a lattice also. If A is a subset of E, and if $s_A(i)$ is the probability that the random walk starting from i, with transition probability matrix Π, will enter A and remain there indefinitely, s_A is a sojourn solution, and every sojourn solution can be obtained in this way. (A sojourn solution is an element of $\mathfrak{P}$ for which the limit along successive random walk steps from each initial point has only the possible values 0 and 1.) The author obtains the usual decomposition of E into recurrent and transient states as an application of his study of sojourn solutions. The boundary $\mathfrak{B}$ is the set of maximum ideals in the lattice of sojourn solutions, and $E \cup B$ is topologized in such a way that each element $\mathfrak{P}$, considered as a function on E, has continuous boundary values. Each such element can be approximated uniformly by linear combinations of sojourn solutions.

If z is a strictly positive element of $\mathfrak{M}$, and if Π' is defined by $\Pi'(i, j) = \Pi(i, j)z(j)/z(i)$, then, if Π has row sums 1, Π' has the same property, and $x \in \mathfrak{M}$ if and only if $\Pi x' = x'$, for $x'(i) = x(i)/z(i)$. Thus a correspondence between $\mathfrak{M}$ and its analogue for Π' is set up and thereby the above is extended to a larger boundary which includes $\mathfrak{B}$ and every boundary obtained by the above relativization in terms of a specified element of $\mathfrak{M}$. The analogue of this relativization in the context of the first boundary value problem for harmonic functions (when $\mathfrak{M}$ becomes the class of positive harmonic functions on a suitable domain) has been used by Brelot [J. Math. Pures Appl. **35** (1956), 297–335; MR **18**, 296] for a somewhat different purpose. Finally, replacing Π by the transition matrix of a corresponding random walk reversed in time, an adjoint boundary is obtained. *J. L. Doob.*

Feller, William. On boundaries and lateral conditions for the Kolmogorov differential equations. Ann. of Math. (2) **65** (1957), 527–570.

This paper contains a general analytical method of constructing stationary transition probability functions p_{ij}, for Markov processes with countable state space $E = (0, 1, 2, \cdots)$ and continuous parameter $t \geqq 0$, with given finite transition rates $q_{ij} = p_{ij}'(0)$ such that $\sum_j q_{ij} = 0$. In terms of the Laplace transforms $P_\lambda(i, j) = \int_0^\infty e^{-\lambda t} P_{ij}(t)\,dt$ this amounts to constructing non-negative matrices $P_\lambda = (P_\lambda(i, j))$ with row sums $\leqq \lambda^{-1}$, $(\lambda I - Q)P_\lambda = I$ (Laplace transform version of the Kolmogorov backward equations) and $P_\lambda - P_\mu + (\lambda - \mu)P_\lambda P_\mu = 0$ (resolvent equation). The resolvent equation is equivalent to the requirement that P_λ, operating on bounded column vectors, should have its range independent of λ; thus the author can reduce the problem of constructing P_λ to the search for lateral conditions which will describe the range of P_λ (as a subset of the domain of Q). The author first gives a new existence proof for his minimal solution of the Kolmogorov equations [Trans. Amer. Math. Soc. **48** (1940), 488–515; MR **2**, 101]. For the corresponding matrix $P_\lambda^{\min}$, put $\tilde{x}_\lambda(i) = 1 - \sum_j \lambda P_\lambda^{\min}(i, j)$. Then $\tilde{x}_\lambda$ is the maximal element of $A_\lambda = \{x : 0 \leqq x \leqq 1,\ \lambda x = Qx\}$, so that $\lambda P_\lambda^{\min}$ has row sums 1 if and only if $A_\lambda = (0)$, and then the backward equations have a unique solution. When A_λ is non-trivial its structure is studied by writing $\lambda x = Qx$ in the form $x = \Pi_\lambda x$, the sub-stochastic matrix Π_λ being defined by $\Pi_\lambda(i, j) = (1 - \delta_{ij})q_{ij}(\lambda + |q_{ii}|)^{-1}$, and applying the author's theory of boundaries induced by non-negative matrices [see the paper reviewed above]. It turns out that the "exit boundary" B and the topology for $E \cup B$, induced by Π_λ, are independent of λ, and that vectors v in the range of $P_\lambda^{\min}$ are characterised by the lateral condition: $v(i) \to 0$ as $i \to B$. For other solutions P_λ, each column of $\lambda(P_\lambda - P_\lambda^{\min})$ must belong to A_λ; the simplest solutions are obtained by making each column a multiple of $\tilde{x}_\lambda$, so that $P_\lambda(i, j) = P_\lambda^{\min}(i, j) + \tilde{x}_\lambda(i)y_\lambda(j)$, y_λ being chosen so as to make the range of P_λ independent of λ. Two possible choices of y_λ are treated, (i) y_λ proportional to $\alpha P_\lambda^{\min}$, where $\alpha \geqq 0$ is such that $\sum \alpha(i)(1 - \tilde{x}_\lambda(i)) < \infty$ and (ii) y_λ a solution of $\lambda y = yQ$; the lateral condition in each case has the form $m \lim_{i \to B} v(i) + f(v) = 0$ where $m \geqq 0$ and f is an unbounded linear functional which resembles a derivative at the boundary. In case (ii), the forward equations $P_\lambda(\lambda I - Q) = I$ hold; in case (i), they do not. These two constructions can be generalized by adjoining another state ω to E, and for the resulting processes on the enlarged state space, ω can be an instantaneous state ($q_{\omega\omega} = -\infty$). Finally, it is shown how to construct P_λ satisfying both Kolmogorov equations, particularly when the exit boundary contains only finitely many points. Here P_λ can also be described by specifying its range when operating on row vectors w such that $\sum |w(j)| < \infty$; the corresponding lateral conditions involve the "entrance boundary" induced by non-negative solutions of $\lambda y = yQ$. {Reviewer's remark: The paper contains some computational errors and a few obscurities. These have not been detailed here because the author has informed the reviewer that a forthcoming note in Ann. Math. will contain the necessary corrections and explanations.}

G. E. H. Reuter (Manchester).

Parasiouk, O. S. Sur le problème de la filtration des processus stationnaires généralisés. Ukrain. Mat. Ž. **9** (1957), 210–214. (Russian. French summary)

Let $\{[m(t), n(t)],\ -\infty < t < \infty\}$ be a stationary process. Let $\phi(t) = m(t) + n(t)$ and define $x(t)$ by $x(t) = \int_{-\infty}^{\infty} \phi(t - \tau)k(\tau)\,d\tau$. The function k is to be chosen to make $x(t)$ have a minimal mean square deviation from a specified stationary process. In the present paper, this standard problem is generalized to allow the $n(t)$ (noise) process to be a generalized stationary process in the sense of Gelfand [Dokl. Akad. Nauk SSSR (N.S.) **100** (1955), 853–856; MR **16**, 938] and the solution is reduced to that of a problem

solved by the author in an earlier paper [same Dokl. (N.S.) **110** (1956), 957–958; MR **18**, 657]. *J. L. Doob.*

Koronkevič, A. I. Some remarks on evaluating the accuracy of linear extrapolation and filtration. Teor. Veroyatnost. i Primenen. **2** (1957), 116–121. (Russian. English summary)

The author discusses extrapolation and filtering using relative mean square errors rather than absolute ones.

J. L. Doob (Urbana, Ill.).

Silverman, R. A. Locally stationary random processes. Div. Electromag. Res., Inst. Math. Sci., New York Univ., Res. Rep. No. **MME-2** (1957), i+8 pp.

Let $x(t)$ be a random process with mean zero. Then by the covariance of $x(t)$ is meant the quantity $\Gamma(t, t')=\langle x(t)x^*(t')\rangle$, where the angular parentheses denote the ensemble average and the asterisk denotes the complex conjugate. If this covariance can be written in the form

$$\Gamma(t, t')=\langle|x(\tfrac{1}{2}t+\tfrac{1}{2}t')|^2\rangle R(t-t'),$$

where $R(t)$ is a normalized correlation function, then the process is said to be locally stationary. The author shows that the Fourier transform of $\Gamma(t, t')$, namely the two dimensional spectral density, is in this case also factorable into a non-negative function and the Fourier transform of a non-negative function. *R. S. Phillips.*

Mihoc, G. Sur les lois limites des variable vectorielles enchaînées au sens de Markoff. Teor. Veroyatnost. i Primenen. **1** (1956), 103–112. (Russian summary)

Let $v^h=(v_1^h, v_2^h, \cdots, v_s^h)$ $(h=1, 2, \cdots, m)$ be m vectors in s-dimensional Euclidean space and let $v(1)$, $v(2)$, $\cdots$ be a stationary Markov chain, i.e., $P\{v(t+1)=v^k|v(t)=v^h\}=p_{hk}$, where $p_{hk}\geqq 0$, $\sum_{k=1}^{m} p_{hk}=1$. Using characteristic functions, and under some assumptions concerning the transition matrix (p_{hk}), the author finds the asymptotic distribution of $w(t)=v(1)+v(2)+\cdots+v(t)$ (suitably normalized). The case of unitary vectors is contained in the earlier work of Kolmogorov [Izv. Akad. Nauk SSSR. Ser. Mat. **13** (1949), 281–300; MR **11**, 119] and Siraždinov [Dokl. Akad. Nauk. SSSR (N.S.) **84** (1952), 1143–1146; MR **14**, 187]. *M. D. Donsker.*

★**Lévy, Paul. Brownian motion depending on n parameters: the particular case $n=5$.** Applied probability. Proceedings of Symposia in Applied Mathematics, Vol. VII, pp. 1–20. McGraw-Hill Book Co., New York-Toronto-London, for the American Mathematical Society, Providence, R. I., 1957. $5.00.

Let A, B be points of n-dimensional Euclidean space, and let $X(A)-X(B)$ be a Gaussian random variable with mean 0 and variance the distance between A and B. Let $M(t)$ be the average value of $X(A)-X(0)$ as A varies on the surface of the sphere with center 0 and radius t. The author studies the $M(t)$ process in detail in the case $n=5$. Almost all sample functions have two continuous derivatives, and $[M(t), M'(t), M''(t)]$ defines a Markov process. The $M(t)$ process satisfies two stochastic differential equations, one determining extrapolation (that is, conditional distributions) to the right, the other to the left. The $M(t)$ process is closely related to the process defined by the integral $\Psi(t)=\int_0^t(\lambda t+\mu u)\,dx(u)$, where the $x(u)$ process is the one dimensional Brownian motion process. The $\Psi(t)$ process covariance function is computed, as are the conditional distributions involved in extrapolation to right and left. [For the author's preliminary statements, see C. R. Acad. Sci. Paris **239** (1954), 1181–1183, 1584–1585; MR **16**, 495, and for a continuation of this work see the paper reviewed below.]

J. L. Doob (Urbana, Ill.).

★**Lévy, Paul. A special problem of Brownian motion, and a general theory of Gaussian random functions.** Proceedings of the Third Berkeley Symposium on Mathematical Statistics and Probability, 1954–1955, vol. II, pp. 133–175. University of California Press, Berkeley and Los Angeles, 1956. $6.50.

The general theory, discussed in § 4, is the representation of a Gaussian random function $\phi(t)$, $0\leq t<\infty$, in the form $\int_0^t F(t, u)X(du)$, where X is a Gaussian process with independent increments and $F(t, u)$ is a σ-function of u [see the paper reviewed above]. The representation is said to be canonical if

$$E\{\phi(t)|\phi(u) \text{ for } 0\leq u\leq s\}=\int_0^s F(t, u)X(du)\quad(0\leq s<t).$$

The author makes plausible the existence and essential uniqueness of a canonical representation, provided $\phi(t)$ satisfies certain requirements of separability and continuity; he treats the matter again in more detail in Ann. École Norm. Sup. (3) **63** (1956), 121–156. The usefulness of the canonical representation (as well as the existence of uncanonical ones) is illustrated by the example treated in §§ 2, 3, 5. Let E be Euclidean n-space (or Hilbert space, for $n=\omega$); let $(X(A))_{A\in E}$ be a Gaussian family of random variables such that $X(A)-X(B)$ has mean zero and variance the squared distance AB^2; let $\bar{M}(t)$ be the mean of $X(A)$ over the sphere with center O and radius t; and set $M(t)=\bar{M}(t)-X(O)$. The covariance and the canonical representation of $M(t)$ are found explicitly for n odd; uncanonical representations are exhibited for $n=5, 7$; and the problem of continuing $M(t)$ to the right or left is treated — it is proved, for example, that $M(t)$ is analytic if $n=\omega$. These sections are the elaboration of earlier notes [C. R. Acad. Sci. Paris **239** (1954), 1181–1183, 1584–1585; **240** (1955), 1043–1044; MR **16**, 495, 939]. The results are applied in § 6 in studying $X(A)$, espacially for $n=\omega$.

G. A. Hunt.

Romaní, José. A model of the theory of queues with a variable number of channels. Trabajos Estadist. **8** (1957), 175–189. (Spanish. English summary)

The queuing model in question is the following: arrivals are individually and collectively at random, each of the servers (whose number varies as described below) has the same exponential distribution of service time, and service is in order of arrival with no defections from the queue. Whenever the queue is of size M, a new arrival occasions the addition of a new server, who remains in attendance until the queue has vanished, at which time all but one server are removed. The stationary probabilities $p(r, s)$ of r waiting and s serving are completely determined, and it is shown that the expected number of servers is the ratio of arrival and departure densities. The model includes for examples the following two well known cases: (i) $M=0$, the many-server problem (with s the initial number of servers) for the "busy-signal" system, that is, with no queue possible, and (ii) $M=\infty$, the same problem, but with no limitation on the queue length. *J. Riordan* (New York, N.Y.).

Mullen, James A.; and Middleton, David. The rectification of non-Gaussian noise. Quart. Appl. Math. **15** (1958), 395–419.

The types of noise functions considered are of the type $V_N(t)=\sum_j v_j(t-t_j')$, where $\{t_j'\}$ are the instants of a Poisson process of density γ, and $v_j(t)=h(t)\cos(\omega_0 t+\alpha_j)$, $\omega_0 \gg 1$, where either (i) $h(t)=e^{-t}$, $t>0$, (ii) $h(t)=e^{-t^2}$, or (iii) $h(t)=1$, $0<t<1$. The stochastic process of interest is $I(t)=g[V_N(t)+A_0\cos\omega_0 t]$, where $g(V)=\beta\{\max(0, V)\}^\nu$, $\nu>0$; the magnitude A_0 corresponds to the signal strength.

The authors compute a number of approximations to the correlation function of $I(t)$, corresponding to low and high γ, and low and high signal strengths. Their expressions become more tractable when $\nu=1$ or 2, but even then are quite complicated. A number of the formulas are represented graphically, and a qualitative discussion of the results is given. *E. Reich* (Minneapolis, Minn.).

Uematu, Tosio. On the traffic control at an intersection controlled by the repeated fixed-cycle traffic lights. Ann. Inst. Statist. Math., Tokyo **9** (1958), 87–107.

For a fixed cycle traffic light, let X_n be the length of the queue for south to north traffic at the end of the nth red period for the $S-N$ lane. Let Y_n be the length of the queue for the west to east traffic at the end of the $(n+1)$th red period in the $W-E$ lane. Traffic going W to S or E to W is not mentioned. The accumulated arrivals in the two lanes are assumed to be independent, each defining a process with independent stationary increments. The X_n and Y_n then form two discrete Markov chains whose transition probabilities depend upon the stochastic properties of the arrivals and the lengths of the red and green periods of the traffic cycle.

One selects numbers G and H which are to represent the longest queues one would care to have in each of the two lanes (called "levels of confusion"). Starting essentially from a state of empty queues in both lanes at time zero, the author computes the expected first passage time for the queue lengths to reach G and H respectively as a function of the cycle times and flows. It is now proposed that, as a criterion for the optimal setting of the light cycle, one should choose the ratio of red to green periods in such a way that these two expected first passage times are equal.

The author derives all the relevant formulae, particularly for the case of Poisson arrivals, and gives a numerical example. Although the problem represents an interesting application of probability theory and is solved in a mathematically correct and systematic way, the paper contains no references to earlier work either on the traffic light problems or on the theory of Markov chains, despite the fact that it gives proofs of several theorems dealing with the first passage time probabilities all of which are either well-known or rather simple modifications of well-known theorems. *G. Newell* (Providence, R.I.).

See also: Linear Algebra: Whittle. **Functions of Complex Variables:** Perry. **Banach Spaces, Banach Algebras, Hilbert Spaces:** Umegaki; Nakamura and Umegaki. **Statistics:** Derman. **Biology and Sociology:** Marchand; Kemeny and Snell.

STATISTICS

★**Risser, R.; et Traynard, C.-E. Les principes de la statistique mathématique. Livre II: Corrélation.** Séries chronologiques. 2me éd., revue et augmentée. Traité du calcul des probabilités et de ses applications. Tome I, fasc. IV. Gauthier-Villars, Paris, 1958. xi+418 pp. 7000 francs.

This, the second volume, treats correlation, various correlation indices, the multivariate normal distribution, probability surfaces, and time series. The point of view is definitely that of descriptive statistics and is decidedly pre-Fisherian in attitude. The last chapter, on time series and stationary processes, is up-to-date as of the time of writing, although even here some of the older material could be culled. *L. A. Aroian* (Culver City, Calif.).

★**Schmetterer, L. Grundlagen der Mathematischen Statistik.** VEB Deutscher Verlag der Wissenschaften, Berlin, 1957. 14 pp. DM 1.60.

A popular, historical lecture, delivered at the University of Hamburg, with a critique of various theories.

Linnik, Yu. V. A remark on Cramer's theorem on the decomposition of the normal law. Teor. Veroyatnost. i Primenen. **1** (1956), 479–480. (Russian. English summary)

H. Cramer's theorem is shown to be an equivalent of a particular case of Skitovič-Darmois' theorem on the independence of linear statistics. This means that each one of these theorems can be deduced from the other by a short elementary argument. *Author's summary.*

Jaeckel, K. Hauptachsentransformation der quadratischen Form für die Streuung. Z. Angew. Math. Mech. **37** (1957), 403–404.

A variant of the well-known proof by matrix methods of the independence of sample-mean and sample-variance for a normal population. *Z. W. Birnbaum.*

Oderfeld, J. On the concentration of distribution. Zastos. Mat. **3** (1957), 182–190. (Polish. Russian and English summaries)

Let ϕ be the probability density of a continuous one-dimensional random variable X defined in an interval S. H. Steinhaus suggested the number $w(\phi)=[\int_S \phi^2(x)dx]^{\frac{1}{2}}$ as a measure of the concentration of X about its modes and conjectured properties which the author proves in the following form: (A) If ϕ_1 and ϕ_2 are probability densities, $0\leq c\leq 1$ is a constant and $\phi=c\phi_1+(1-c)\phi_2$, then $w(\phi)\leq cw(\phi_1)+(1-c)w(\phi_2)$. Hence $w(\phi)\leq\max[w(\phi_1), w(\phi_2)]$. (B) If v_i^2 is the integral of ϕ^2 over the ith of N subintervals into which S is partitioned, then $\sum_{i=1}^N v_i$ is maximal for $v_1=v_2=\cdots=v_N$. Statement (A) immediately follows from the inequality of Schwarz, statement (B) from the familiar differentiation criterion for extrema. A few examples of applications to practical situations requiring high concentration of X are given and the connection between $w(\phi)$ and the standard deviation is discussed. *H. M. Schaerf* (Madison, Wis.).

Linnik, Yu. V. On "determining" statistics; a generalization of the problem of moments. Dokl. Akad. Nauk SSSR (N.S.) **113** (1957), 974–976. (Russian)

Let X be a random variable with a law of distribution

$F(x)=P(X<x)$ and $\xi=(X_1, \cdots, X_n)$ be a random vector with independent components each determined by the law $F(x)$. Further, let $Q(\xi)$ be a continuous statistic. The author poses the following general question: In which cases does a knowledge of the distribution $F_Q(x)=P(Q<x)$ allow us to uniquely determine the distribution $F(x)$? If for a given statistic this is possible in a class of distributions R, then the statistic is called "determining" for the given class R. The author remarks that this problem may be considered as a generalization of the classical (determined) moment problem.

In this paper the author limits himself to what he calls "definite" statistics, the essential conditions on these being that they are homogeneous and the level surfaces $Q(\xi)=c$ are bounded and piecewise-smooth. He also limits himself to probability densities which lie in a class of type (D_p). A set of continuous functions defined on $[a, b]$ form a class of type (D) if the difference of any two functions is either identically zero or has at most a finite number of zeros. A set of functions form a class of type (D_p) on $[-\alpha, \alpha]$ $(\alpha \geqq \infty)$ if each bounded segment $[a, b] \subset [-\alpha, \alpha]$ may be decomposed into a finite number of segments on each of which the given set of functions belongs to a class of type (D).

Four theorems are stated without proof, of which the following is representative: A definite statistic is determining in a class of even continuous probability densities of type (D_p) given on the segment $[-\alpha, \alpha]$, each having there at most a countable number of zeros.

The proofs of two of the theorems depend on the following lemma, which seems to be of independent interest: If $g(x)$ and $h(x)$ are even positive definite functions, $h(x)$ is holomorphic in a neighborhood of the origin and the origin is a limit of zeros of $g(x)-h(x)$, then $g(x) \equiv h(x)$ on the whole axis. *A. Devinatz* (St. Louis, Mo.).

Breny, H. Sur quelques problèmes d'analyse statistique posés par la physique des microcorpuscules. I. Distributions de Poisson et mesures relatives. Ann. Soc. Sci. Bruxelles. Sér. I. **71** (1957), 135–160.

For $i=1, 2, \cdots, M$ the symbol n_i represents a Poisson variable, independent of all the others, with its expectation equal to $\alpha T_i f_i(\theta_1, \theta_2, \cdots, \theta_r)$, where $r \leqq N$. The factors T_i are supposed known. The factor α may be known or not, but is not interesting. The θ's are parameters to be estimated. For several specialized forms of the functions f_i and for the assumption $\sum n_i=N$, where N is fixed and greater than zero, the author studies several estimators of the parameters θ provided by the methods of maximum likelihood, of minimum χ^2, of minimum modified χ^2 and of least squares. *J. Neyman.*

Cohen, A. Clifford, Jr. Restriction and selection in multinormal distributions. Ann. Math. Statist. **28** (1957), 731–741.

The author derives the maximum likelihood estimates for the parameters of a multinormal distribution based upon samples from the distribution restricted, truncated, or selected through the first coordinate. Some numerical illustrations are given. *F. C. Andrews* (Eugene, Oreg.).

Guttman, Irwin. On the power of optimum tolerance regions when sampling from normal distributions. Ann. Math. Statist. **28** (1957), 773–778.

"Merit" for normal variable tolerance regions was defined by Fraser and Guttman [same Ann. **27** (1956), 162–179: MR **17**, 871]. The associated tables are presented here. *I. R. Savage* (Minneapolis, Minn.).

Roy, S. N.; and Gnanadesikan, R. Further contributions t multivariate confidence bounds. Biometrika **44** (1957), 399–410.

This paper continues previous work of Roy [Ann. Math. Statist. **25** (1954), 752–761; **27** (1956), 856–858; MR **16**, 382; **18**, 772]. There are worked out a number of confidence bounds on the characteristic roots connected with (i) one population dispersion matrix, (ii) two population dispersion matrices, (iii) the regression matrix of a set of p variates on a set of q variates, and (iv) the multivariate linear hypothesis on population means. In each case one gets further bounds which hold simultaneously with the same confidence when any subset of the original variates and the corresponding dispersion matrices, etc., are considered. Numerical examples illustrate the use of the techniques presented, especially for (iv) and (iii).

S. W. Nash (Vancouver, B.C.).

Ura, Shoji. On Scheffé's analysis of variance for paired comparisons. Rep. Statist. Appl. Res. Un. Jap. Sci. Engrs. **4** (1957), 132–146.

A modified model is proposed for the same experimental design as that discussed by Scheffé [J. Amer. Statist. Assoc. **47** (1952), 381–400; MR **14**, 488]. In a paired comparison test of t brands, each possible pair is compared by $2r$ judges, r for each order of presentation. Scheffé supposed that the scores could be represented by $2n+1$, e.g. 7, successive integers. The present author supposes that degrees of preference are normal variates. He finds the relations between the parameters of his and Scheffé's models. He then considers the type of scoring system needed to maximize the power of the analysis of variance test and to minimize the bias in estimating parameters, assuming the new model holds.

S. W. Nash (Vancouver, B.C.).

Cornish, E. A. An application of the Kronecker product of matrices in multiple regression. Biometrics **13** (1957), 19–27.

Chakravarti, I. M. On a relation between canonical correlations and partial canonical correlations. Calcutta Statist. Assoc. Bull. **5** (1954), 185–187.

Watson, G. S. The χ^2 goodness-of-fit test for normal distributions. Biometrika **44** (1957), 336–348.

A chi-square is formed using k groupings, and the expected frequencies in the cells depend on s parameters, which are to be estimated. If the parameters are estimated from the ungrouped original data, then the calculated chi-square is $\chi_{k-s-1}^2+\lambda_1 y_1^2+\cdots+\lambda_s y_s^2$, where $\lambda_1, \cdots, \lambda_s$ are certain characteristic roots, all in the interval $(0, 1)$, and $y_1, \cdots, y_s$ are standard normal variates, independent of each other and of χ_{k-s-1}^2. In general, $\lambda_1, \cdots, \lambda_s$ depend on the s parameters. However, if the expected cell frequencies are prescribed, if the population to be fitted is normal, and if the cell boundaries are determined using the sample mean and variance, then to the order of approximation used, λ_1 and λ_2 are independent of population mean and variance. They can be calculated explicitly and tend to zero rapidly as k increases.

S. W. Nash (Vancouver, B.C.).

Cochran, William G. Analysis of covariance: Its nature and uses. Biometrics **13** (1957), 261–281.

This is the introduction to the other papers in the special issue of Biometrics on covariance, and presents

both a lucid discussion of the nature and principal uses of the analysis of covariance and a description of standard analytical methods and tests of significance. The principal uses listed are the increase of precision in randomized experiments, the removal of the effects of disturbing variables in observational studies, the illumination of the nature of treatment effects, the fitting of regressions in multiple classifications, and the analysis of data when some observations are missing. The nature of the covariance adjustment, the theory underlying covariance analyses, and the assumptions required for covariance analyses are also discussed to some extent. Some other topics are discussed briefly. The article is an excellent introduction to the topic of covariance as well as to the articles which follow in the covariance issue.

W. T. Federer (Ithaca, N.Y.).

Roy, J. On some tests of significance in samples from bipolar normal distributions. Sankhyā **14** (1954), 203–210.

Weiss, Lionel. A note on confidence sets for random variables. Ann. Math. Statist. **26** (1955), 142–144.

Barton, D. E. Neyman's Ψ_k^2 test of goodness of fit when the null hypothesis is composite. Skand. Aktuarietidskr. **39** (1956), 216–245 (1957).

The author considers the following problem: Let $X_1, \cdots, X_n$ be independent random variables with continuous cumulative distribution function $F(X|\theta)$, where θ is a p-dimensional vector. If we form $\psi_k^2 = \sum_{i=1}^{k} [n^{-1/2}\sum_{t=1}^{n} \pi_r(F(X_t|\theta))]^2$, we obtain Neyman's smooth test. The author previously [Skand. Aktuarietidskr. **36** (1953), 24–63; MR **15**, 453] considered the effect of grouping on ψ_k^2. Here he considers the effects, both of grouping and of estimating θ, on the distribution of ψ_k^2 under the null hypothesis. If suitable regularity conditions are satisfied and the grouping is made independently of the observations (pregrouping), the limiting distribution of ψ_k^2 under the null hypothesis is that of $\sum_{i=1}^{k} y_i^2$, where (if $k>2p$) the y's are normal, one y may have non-zero mean, and $2p$ other y's may have variance not unity. If maximum likelihood or equivalent estimates are used and the usual regularity conditions are satisfied, only p y's may have variance different from 1, and the variances are between 0 and 1. If the observations are first transformed and the transforms grouped, the only difference is that the term with non-zero mean does not occur. If no grouping is made, this last also holds.

The results are similar to those of Chernoff and Lehman [Ann. Math. Statist. **25** (1954), 579–586; MR **16**, 384].

A detailed treatment of postgrouping for the χ^2 case may be found in A. R. Roy's unpublished doctoral dissertation, in which he successfully treats the analytic difficulties to which the author alludes. *H. Rubin.*

Derman, Cyrus. Non-parametric up-and-down experimentation. Ann. Math. Statist. **28** (1957), 795–798.

Let $F(x)$ be a distribution function, and for each x let $Y(x)$ be a 0-1 random variable such that $P[Y(x)=1]=F(x)$. An estimate is desired for θ such that $F(\theta-0)\leq\alpha\leq F(\theta)$, where α is a fixed number, $\frac{1}{2}\leq\alpha<1$ (obvious modifications arise if $0<\alpha<\frac{1}{2}$). A nonparametric scheme due to Robbins and Monroe [same Ann. **22** (1951), 400–407; MR **13**, 144] can be applied when x ranges over the real line. The author gives a procedure for the case where x takes the values $0, \pm1, \pm2, \cdots$ (obvious modifications occur when x takes any equally spaced set of values). Let x_1 be an arbitrary integer and define $\{x_n\}$ recursively: by $x_n = x_{n-1}-1$ or $x_{n-1}+1$, with respective probabilities $1/(2\alpha)$ and $1-1/(2\alpha)$ if $y_{n-1}=1$; and by $x_n=x_{n-1}+1$ with probability 1 if $y_{n-1}=0$, where y_n is the value obtained for $Y(x_n)$. Then $\{x_n\}$ is a Markov chain. Let θ_n be the estimate of θ based on n observations and let it be defined as the most frequent value of x, if unique, or the arithmetic average of the most frequent values, if not unique. Theorem: If F is strictly increasing for $\theta-1\leq x\leq\theta+1$, then $P[\max(|\limsup_{n\to\infty}\theta_n-\theta|, |\liminf_{n\to\infty}\theta_n-\theta|)<1] = 1$. The proof depends upon a study of the properties of the stationary distribution of the Markov chain.

T. E. Harris (Santa Monica, Calif.).

Sakaguchi, Minoru. Notes on statistical applications of the information theory. III. Rep. Statist. Appl. Res. Un. Jap. Sci. Engrs. **5** (1957), 9–16.

[For parts I–II see same Rep. **1** (1952), no. 4, 27–31; **4** (1955), 57–68; MR **14**, 996; **17**, 758.]

The author formulates an experiment as a communication system with noise. Following Lindley [Ann. Math. Statist. **27** (1956), 986–1005; MR **18**, 783], he defines the amount of information provided by an experiment with prior knowledge, by Shannon's formula for the transmission rate of a noisy channel [Bell System Tech. J. **27** (1948), 379–423, 623–656; MR **10**, 133]. The amount of information is a concave functional of the prior density and a convex functional of the conditional density of the observation, corresponding to a mixture of prior knowledge, or a mixture of experiments respectively. The author defines the capacity of an experiment similarly to the capacity of a channel, that is, the maximum of the rate for all possible prior densities. He considers, in particular, properties of the capacity for a finite input space (the maximum over the space of all n-dimensional probability vectors). The author also discusses the relations between the methods of comparing two dichotomous experiments given by Blackwell [Proc. 2nd Berkeley Symposium Math. Statist. and Probability, 1950, Univ. of California Press, 1951, pp. 93–102; MR **13**, 667], Bradt and Karlin [Ann. Math. Statist. **27** (1956), 390–409; MR **19**, 332], and Lindley. In particular, he shows that experiments having uniformly smaller Bayes risks for the corresponding decision problems are more informative according to Lindley. This is also true for the definition of more informative according to Bradt and Karlin. A necessary and sufficient condition for one dichotomous experiment to have a uniformly smaller Bayes risk than another is that the first be more informative according to Blackwell. *S. Kullback* (Washington, D.C.).

Kramer, Clyde Young; and Bradley, Ralph Allan. Examples of intra-block analysis for factorials in group divisible, partially balanced, incomplete block designs. Biometrics **13** (1957), 197–224.

Cox, D. R. The use of a concomitant variable in selecting an experimental design. Biometrika **44** (1957), 150–158.

Rao, C. Radhakrishna. On the recovery of inter block information in varietal trials. Sankhyā **17** (1956), 105–114.

Isbell, J. R.; and Wagner, F. J. Military evaluation and statistical decision (u). Ballistic Research Laboratories, Aberdeen Proving Ground, Md., Rep. no. 1014, 34 pp. (1956). (Government Agencies, their contractors and others cooperating in Government research may obtain reports directly from the Ballistic Research Laboratories. All others may purchase photographic copies from the Office of Technical Services, Department of Commerce, Washington 25, D. C.)

A general discussion of statistical decision theory containing many ideas and arguments. Among other things, four decision procedures are criticised along the lines of J. W. Milnor [Thrall, Coombs and Davis (ed.), Decision processes, Wiley, New York, 1954, pp. 49–59; MR **16**, 605]. The four procedures are the Laplace equiprobable procedure, the Wald and Savage forms of the minimax procedure, and the Hurewicz degree-of-optimism procedure. A fifth procedure is then suggested that is less open to theoretical criticism, but which is at present difficult to apply in practice. It is essentially the reviewer's "Type II minimax" procedure (higher types are conceivable), independently suggested by Hurewicz under the name "Bayes minimax" procedure [Econometrica **19** (1951), 343–344; I. J. Good, J. Roy. Statist. Soc. Ser. B. **14** (1952), 107–114; **17** (1955), 195–196; MR **17**, 981]. In a note attached to the review copy, one of the authors writes that he should have noted the last of these references, and he acknowledges L. J. Savage for pointing out this omission. *I. J. Good* (Cheltenham).

Des Raj. On estimating parametric functions in stratified sampling designs. Sankhyā **17** (1957), 361–366.

The problem of the estimation of a number of linear functions of the means of the strata in a stratified sampling design is considered. The author obtains equations for the optimum allocation of the sample size in several cases corresponding to the result to be achieved by the sample. An example is given to illustrate these cases. *O. P. Aggarwal* (Edmonton, Alta).

Saito, Kinichiro. Some results in the theory of sampling on successive occasions with partial replacement of units. Rep. Statist. Appl. Res. Un. Jap. Sci. Engrs. **4** (1957), 125–131.

Consider a population π and a set of times $t_1, t_2, \cdots, t_h$. Each element of the population has a set of h values associated with it, one for each time t_i. Let Y_i be the random variable representing the population values at time t_i. Denote $E(Y_i)=\mu_i$, var $(Y_i)=\sigma^2$ (where σ is assumed independent of time). It is also assumed that the covariance $(Y_i, Y_j)=\rho^{(i-j)}\sigma^2$.

Further, suppose a series of samples have been drawn from the population with some old elements being replaced by new elements at each successive occasion. This is known as "rotation sampling" or "sampling with partial replacement of units". The best linear unbiased estimate of μ_i, which has been given by Patterson [J. Roy. Statist. Soc. Ser. B. **12** (1950), 241–255], depends upon ρ. The present paper reports some results of the effect on the estimate and its variance of an estimation of ρ by the sample correlation coefficient r. The effect of failure to hold of the exponential correlation condition is also studied. Patterson's theorem is used further to obtain results in the case of two stage sampling. *D. G. Chapman.*

Rangarajan, R. A note on two stage sampling. Sankhyā **17** (1957), 373–376.

Consider a two stage sampling plan in which the statistician decides in advance of the first stage, for each primary unit i, how many observations m_i will be made there if the unit is selected. The total number m of observations becomes a random variable but the optimum allocation of the m_i yields estimates with smaller variance than that obtained from the optimum allocation of the m_i within the primary units actually sampled and with the total number of observations fixed at the $E(m)$ of the first procedure. This result, when the selection of primary units is random is due to Godambe [J. Roy. Statist. Soc. Ser. B. **13** (1951), 216–218; MR **14**, 298]. The present paper extends this to the case where sampling of the primary units takes place with arbitrary probabilities.

D. G. Chapman (Seattle, Wash.).

Shiskin, Julius; and Eisenpress, Harry. Seasonal adjustments by electronic computer methods. J. Amer. Statist. Assoc. **52** (1957), 415–449.

Electronic computer methods are utilized to apply two different procedures for the elimination of the seasonal component of economic time series. The first, simpler method uses moving averages and ratios; the second uses also a control chart procedure for the ratios. Many results are presented graphically and possible improvements of the methods are discussed. An example is worked: Total unemployment in the United States, March 1940 to April 1957. *G. Tintner* (Ames, Iowa).

★ **Kiveliovitch, M.; et Vialar, J. Les séries chronologiques et la théorie du hazard.** Publ. Sci. Tech. Ministère de l'Air, Notes Tech. no. **65**, Paris, 1957. xii+129 pp. 1650 francs.

The probability distribution, mean values and variances of maxima and minima, the probability distributions of runs and various tests are developed for the analysis of time series. The tests are large sample tests, based upon the normal approximation, and use confidence limits. Discrete uniform distributions, arbitrary discrete distributions, continuous distributions and the normal distribution are used for the underlying populations There are applications to metereological data. *G. Tintner.*

Geisser, Seymour. The distribution of the ratios of certain quadratic forms in time series. Ann. Math. Statist. **28** (1957), 724–730.

The Durbin-Watson procedure of omitting a term in the numerator of the sample serial correlation coefficient (R) in order to obtain double roots is extended to that of Von Neumann's $\mathcal{N}$ statistic. The non-null distribution of the Durbin-Watson modified R and the modified $\mathcal{N}$ are then derived. Certain moments are also presented.

R. L. Anderson (Raleigh, N.C.).

Wagner, Harvey M. A Monte Carlo study of estimates of simultaneous linear structural equations. Econometrica **26** (1958), 117–133.

The author computes the empirical distribution of various statistics from 100 sets of observations for time series of length 20. He considers the least square estimates, limited information single equation (L.I.S.E.) estimates, and instrumental variables estimates. He shows that the bias of the least squares estimates is quite apparent, occurring in most samples, and that the other estimates appear unbiased. The various estimates have approximately the same root mean square error. Further-

more, the L.I.S.E. estimates of the parameters and of the variance of the residuals seem to have the distributions one would expect from medium sample theory.

H. Rubin (Eugene, Oreg.).

Zubrzycki, S. On estimating gangue parameters. Zastos. Mat. **3** (1957), 105–153. (Polish. Russian and English summaries)

Let p be a point in the plane, and $y(p)$ a real-valued random variable corresponding to this point, which represents the value of one of the parameters of a geological deposit such as concentration of a valuable metal, contents of impurities, etc. The author assumes that $y(p)$ is a stochastic process in the plane such that the expectation $E[y(p)]$ and the variance $\sigma^2[y(p)]$ are independent of p, and the correlation function $R(p, p')$ depends only on the distance between p and p'. He then discusses such problems as the estimation of $E[y(p)]$, given measurements y_i^* of $y(p_i)$, $i=1, 2, \cdots, k$, where these measurements in turn are subject to random errors of observation. Numerical illustrations are included. *Z. W. Birnbaum.*

See also: Biology and Sociology: Bailey.

PHYSICAL APPLICATIONS

★ **Fischer, Otto F. Five mathematical structural models.** Axion Institute, Lidingö, Stockholm, 1957. vi+412pp. $10.00.

In this book the author discusses the application of the theory of quaternions to five physical theories which were discussed in lesser detail in "Universal mechanics and Hamilton's quaternions" [Axion Inst., Stockholm, 1951; MR 13, 502]. An eight page discussion of quaternions and a fifteen page summary of vectors and dyadics in three space are given in an attempt to make this book self-contained and independent of the earlier one.

The five applications involve (1) the theory of electromagnetism, (2) the stress-energy tensor in space-time, (3) von Mises' theory of motors, (4) Dirac's theory and (5) classical wave theory and wave mechanics.

Each of these subjects can be described in four-dimensional space-time, in terms of tensors and spinors. The author prefers to describe them in terms of quaternions, which provide an equivalent description. This equivalence is known and can be established in terms of the connection between the inner automorphisms of the quaternions over the complex field and the complex orthogonal group in three variables, which in turn is isomorphic to the proper Lorentz group in four variables.

The author's notation and presentation are such that the details of this correspondence are not clearly delineated and as a result, the mathematical reader may have some difficulty in following the discussion. *A. H. Taub.*

Mechanics of Particles and Systems

Stehle, P. Dynamical principle for classical mechanics. Amer. J. Phys. **24** (1956), 626 – 629.

The author shows that the difference between the classical and quantum statements of the variational principle is not one between classical and quantum theories, but between two ways of formulating the classical principle, and that a classical theory can be formulated in the same way as are the quantum theories.

E. B. Schieldrop (Oslo).

Havas, P. The range of application of the Lagrange formalism. I. Nuovo Cimento (10) **5** (1957), supplemento, 363–388.

Given a set of equations $G_i(q, \dot{q}, \ddot{q}, t)=0$; $(i=1, 2, \cdots, n)$, Helmholtz and others have established the necessary and sufficient conditions for the existence of a function $L(q, \dot{q}, t)$ such that $G_i=\mathscr{L}[L]$, where $\mathscr{L}[\ ]$ stands for the Lagrangian operator. The author extends the range of applicability of the Lagrange formalism to a set of equations $G_i^\beta=\ddot{q}_i+g_i(q, \dot{q}, t)$, and presents the conditions for the existence of "integrating factors" $f_i(q, \dot{q}, t)$ and functions $L^\beta(q, \dot{q}, t)$ satisfying $f_iG_i^\beta=\mathscr{L}[L^\beta]$. Five illustrative examples are treated. *E. B. Schieldrop* (Oslo).

★ **Седов, Л. И. [Sedov, L. I.] Методы подобия и размерности в механике. [Similarity and dimensional methods in mechanics.]** 4th ed., revised and enlarged. Gosudarstv. Izdat. Tehn.-Teor. Lit., Moscow, 1957. 375 pp. 13.65 rubles.

The present edition differs from the 3rd [MR **17**, 909; the 2nd was reviewed in MR **14**, 809] only by certain additions to chapter two (Similarity, modelling and various examples of dimensional theory) and in a reworking of section 6 of chapter five (Explosion of novae and supernovae).

Kappus, R. Contribution au calcul des matrices de rigidité. Rech. Aéro. no. **52** (1956), 43–49.

Novoselov, V. S. Application of non-linear non-holonomic coordinates in analytical mechanics. Leningrad. Gos. Univ. Uč. Zap. **217**. Ser. Mat. Nauk **31** (1957), 50–83. (Russian)

Novoselov, V. S. Extended equations of motion of non-linear non-holonomic systems. Leningrad. Gos. Univ. Uč. Zap. **217**. Ser. Mat. Nauk **31** (1957), 84–89. (Russian)

The two papers present a very formal discussion of nonholonomic systems subject to restraints not necessarily linear in the velocities. *W. Kaplan.*

★ **Easthope, C. E. Three dimensional dynamics. A vectorial treatment.** Academic Press Inc., New York; Butterworths Scientific Publications, London, 1958. viii+277 pp. $7.80.

There are 9 chapters, with the following titles: vector algebra, motion of a particle, kinematical motion of a rigid body, theory of rotating axes, moments of inertia, equations of motion of a rigid body, Euler's equations and applications, precessional motion, impulsive motion. There is also an appendix on three-dimensional statics. Tensors are not introduced.

Mihailović, Dobrivoje. Verallgemeinerung einiger Ergebnisse im Zweikörperproblem mit veränderlicher Massensumme. Bull. Soc. Math. Phys. Serbie **8** (1956), 195–198. (Serbo-Croatian. German summary)

The author establishes a relation between the areal velocities in the classical problem of two bodies and the corresponding problem with a general law for the variation

of the sum of masses. This law is the author's generalization [same Bull. **5** (1953), no. 1–2, 67–76; MR **16**, 653] of Batyrev's law of variation.

T. P. Anđelić (Belgrade).

Mihailović, Dobrivoje. Über das Energieintegral im Zweikörperproblem mit veränderlicher Massensumme. Bull. Soc. Math. Phys. Serbie **8** (1956), 199–202. (Serbo-Croatian. German summary)

A. A. Batyrev [Astr. Ž. **26** (1949), 56–59; MR **10**, 577] has, by a suitable substitution, reduced the problem of two bodies, whose total mass $M=m_1+m_2$ varies according-ly to the law

$$M=M_0/(1+\alpha t),$$

where α is a positive constant, to the classical problem with invariable masses.

In this paper the author determines a relation between the integral of energy for the problem with variable sum of masses and the integral of energy when masses are invariable. He uses for that purpose Batyrev's transformation as well as a relation between the velocities in the original and transformed motion, which was deduced by the author himself elsewhere [Bull. Soc. Math. Phys. Serbie **4** (1952), no. 3–4, 49–52; MR **15**, 903].

T. P. Anđelić (Belgrade).

Hertig, Ricardo R. New equations for the motion of mechanical systems. An. Soc. Ci. Argentina **164** (1957), 49–57. (Spanish)

The equations of motion given here are the familiar equations derived by Appell. The derivation of the equations differs, at least superficially, from the one usually given, and it possesses some features which may simplify the calculations involved in applying the equations to particular problems. *L. A. MacColl.*

Bentsik, Ettore. Su di un problema del tipo di quello della bussola giroscopica nel caso di un corpo rigido asimmetrico. Rend. Sem. Mat. Univ. Padova **27** (1957), 176–180.

The author considers motions of a rigid body which is subject to the following constraints: The centroid is fixed in a plane π which is itself fixed in the rotating earth, and a line f, which is fixed in the body and passes through the centroid, is constrained to lie in π. Frictional and gravitational effects are neglected. The differential equations of motion are written in vectorial form, and some properties of a few particular motions are discussed.

L. A. MacColl (New York, N.Y.).

Grioli, Giuseppe. Sul moto di un corpo rigido asimmetrico soggetto a forze di potenza nulla. Rend. Sem. Mat. Univ. Padova **27** (1957), 90–102.

The motions discussed may be interpreted concretely as those of an electrically charged asymmetrical rigid body, with its centroid fixed, immersed in a spatially and temporally constant magnetic field. The differential equations of motion are written in vectorial form, and some of their consequences are developed. Two integrals are obtained: the energy integral, and another which depends linearly on the angular momentum of the body. It is shown that in general a rotation, necessarily uniform, about an axis fixed in space and in the body is possible only if the axis is one of the principal axes of inertia. However, if the angular velocity has a certain particular value, the axis may be fixed arbitrarily in the body. Finally, it is shown that motions in which the body precesses regularly are impossible. *L. A. MacColl.*

Ilyin, R. F. On the dynamics of the ascending and the descending branches of a hoist cable. Akad. Nauk Ukraïn. RSR. Prikl. Meh. **3** (1957), 325–335. (Ukrainian. Russian and English summaries)

Klotter, K.; und Kreyszig, E. Über eine besondere Klasse selbsterregter Schwingungen. Ing.-Arch. **25** (1957), 389–403.

The authors consider the class of self-excited vibrations governed by the "modified van der Pol" differential equation

(1) $\ddot{q}-(\operatorname{sgn}\dot{q})\frac{1}{2}\beta(1-\alpha^2q^2)\dot{q}^2+\kappa^2f(q)=0$, with $\beta>0$.

In order to correspond to physical systems, the restoring force $f(q)$ satisfies the conditions: $f(q)\geqq 0$ for $q>0$; $f(q)\leqq 0$ for $q<0$; $f(q)\not\equiv 0$. Substituting $V=\dot{q}^2$, a first-order differential equation is obtained, from which a formal relation between successive maximum displacements in the positive and negative directions is derived. Upper and lower bounds for the amplitude of the limit cycle, Q^*, are investigated. For $f(q)=q^{2m+1}$ ($m=0, 1, \cdots$), an approximate expression for Q^* is derived and compared with exact values for the cases $m=0$ and $m=1$. The method of isoclines is used to find the trajectories and limit cycle corresponding to equation (1). Another approximation for the amplitude of the limit cycle when $f(q)=q$ is obtained by the method of slowly varying amplitude and phase.

G. B. Warburton (Edinburgh).

Ziemba, Stefan. The influence of viscosity damping on the form of the trajectories of free vibration. Arch. Mech. Stos. **9** (1957), 487–504. (Polish and Russian summaries)

After discussing the phase trajectories of free oscillations with linear damping, following the analysis of N. Minorsky [Introduction to non-linear mechanics, Edwards, Ann Arbor, 1947; pp. 16–19; MR **8**, 583], the author considers the phase trajectories of free oscillations with nonlinear damping, represented by: $A\ddot{y}+[B+F(\dot{y})]\dot{y}+Cy=0$, where $A>0$, $C>0$, $B\geqq 0$, $F(-\dot{y})=F(\dot{y})\geqq 0$, $y=y(t)$. He gives a graphical method of constructing these trajectories, stating that this method is more laborious than the method of A. Liénard [Rev. Gén. Elec. **23** (1928), 901–912, 946–954], but illustrates more clearly the influence of damping on the behaviour of the trajectories.

G. B. Warburton (Edinburgh).

Apetaur, Milan; und Půst, Ladislav. Die grafischen Lösungsmethoden der Schwingungsbewegungen der Kraftwagen mit Berücksichtigung der nichtlinearen Federung. Apl. Mat. **2** (1957), 81–104. (Czech. German summary)

Es handelt sich um zwei nichtlineare Differentialgleichungen zweiter Ordnung, welche die Bewegung eines mechanischen oder elektrischen Systems beschreiben. Die Lösungsmethode besteht in der Konstruktion der Tangenten der Integralkurven in den Phasenebenen und geht auf Liénard und Jacobsen [Jacobsen, J. Appl. Mech. **19** (1952), 543–553; MR **14**, 502] zurück. Sie ist insbesondere für das Studium der Übergangserscheinungen in nichtlinearen Systemen geeignet und wird in dem Artikel auf einige Aufgaben der Federung des Kraftwagens angewandt. *M. Zlámal* (Brno).

Toraldo di Francia, Giuliano. Sulla gittata massima di un missile. Atti Accad. Naz. Lincei. Rend. Cl. Sci. Fis. Mat. Nat. (8) **21** (1956), 404–407.

The author considers a missile of variable mass subject to a constant gravitational force, a resistance proportional to the missile velocity, and an extra propulsive force F, of given magnitude. The author proves that the maximum range is attained when F has constant direction. This generalizes a result of Fried and Richardson [J. Appl. Phys. **27** (1956), 955–961; MR **18**, 524] ,who ignored air resistance. The variational problem is solved by the procedure of Volterra [Theory of functionals..., Univ. Central, Fac. Ci., Madrid, 1930, Ch. 1]. *W. Kaplan.*

Gold, Louis. Aspects of high energy ballistics. J. Franklin Inst. **264** (1957), 301–311.

The objective of this brief exposition is the systematic development of the appropriate celestial mechanics pertinent to the exterior ballistics of long-range missiles. Effects due to air-resistance, to the spheroidal form of the Earth, to Coriolis force, to the Earth's rotation, etc., are ignored. The discussion starts with the case of vertical launching [recognized in a later appended footnote, to have been adequately covered by J. H. Jeans in Theoretical mechanics, Ginn, Boston, 1907]. Appended is a table of gunnery data for idealized flight, wherein it appears that the remotest objectives on the Earth are (theoretically) within an hour flight time. The author remarks, "It is a pity in this era of the launching of satellites one still finds antiquated presentations of this phase of physics", to which the reviewer is inclined to reply: "Of course, much scientific work on missiles remains unavailable to the public press."

A. A. Bennett (Providence, R.I.).

See also: Numerical Methods: Gillies and Hunt. **Astronomy:** Radsievsky and Gelfgat.

Statistical Thermodynamics and Mechanics

★ **Wilson, A. H. Thermodynamics and statistical mechanics.** Cambridge University Press, New York, 1957. xv+495 pp. $9.50.

The book on thermodynamics and statistical mechanics written by A. H. Wilson is worthy of attention for a number of reasons: The author of "The theory of metals" [Cambridge, 1936] is well known; books on thermodynamics, although numerous, are seldom good or liked, probably since they never equal Gibbs' presentation nor make the task of understanding easier; and finally, perhaps mathematicians and logicians may yet be persuaded to devote deserved attention to these fields.

Professor Wilson begins with a fairly standard presen tation of the first and second law of thermodynamics including conditions of equilibrium and stability. It seems difficult to avoid circular arguments. On page 6 the author states, "If a body is in equilibrium in a 'vessel' and if its state can only be altered from outside either by action at a distance or by moving the 'wall' of the vessel, the vessel is said to be an adiabatic vessel", and on page 52, "For the equilibrium of any isolated system it is necessary and sufficient that in all possible variations of the state of the system which do not alter its internal energy, volume or the masses of any of its independent constituents, the variation of its entropy shall either vanish or be negative". Chapter 4 is devoted to an alternative but preferred discussion of the laws of thermodynamics according to Caratheodory's principle based on the properties of the Pfaff equation and is a welcome addition.

Statistical mechanics is introduced on a postulatory basis and no discussion of ergodic theory is given. The statistical expressions for thermodynamic functions are derived as most probable values of phase rather than average values (method of steepest descent). Applications discussed include imperfect gases, gas mixtures and the equilibrium of chemical reactions, solutions of non-electrolyte and electrolyte systems, and electrical and magnetic phenomena.

Although written for the theoretical physicist, the book contains a wealth of physical chemistry and will certainly become a much used work. *J. Ross.*

Temperley, H. N. V. Statistical mechanics of non-crossing chains. I. Trans. Faraday Soc. **53** (1957), 1065–1073.

The statistical mechanical studies of long chain molecules (in which crossing configurations are prohibited by steric considerations), the Onsager-Ising lattice, non-linear diffusion problems, and the imperfect gas including condensation, are all limited by the lack of analytical solutions for the three-dimensional restricted random walk problem of emunerating paths with given probability of successive steps and the restriction of no crossing of paths. The article offers the suggestion of an approach based upon Zwanzig's observation that for certain kinds of bodies (rigid squares and cubes) the virial coefficients can be obtained simply from a knowledge of the corresponding one dimensional case and, furthermore, limits can be placed upon these coefficients for bodies with slightly more resemblance to molecules (rigid spheres). Although the article is of interest, "So far, no analytic solutions of non-trivial problems have been obtained by this method, but prospects of doing so seem at least as good as they are for 'equation-of-state' problems". *J. Ross.*

Mason, Edward A. Higher approximations for the transport properties of binary gas mixtures. I. General formulas. J. Chem. Phys. **27** (1957), 75–84.

Higher approximations for the transport properties of binary gas mixtures are derived: (1) by the method of Chapman and Cowling; (2) by an extension of a method due to Kihara. Particular attention is paid to diffusion and thermal diffusion because of their importance in obtaining information on the forces between unlike molecules. The results can be used to test the convergence and accuracy of the theoretical formulas for various molecular models, and to supply higher correction terms if needed. (Author's summary). *J. Ross* (Providence, R.I.).

Bodó, Z. The solution of the Boltzmann equation by assuming a constant relaxation time. Acta Phys. Acad. Sci. Hungar. **8** (1957), 177–179.

The article gives an exact solution of the Boltzmann equation for electrons following the Maxwell-Boltzmann distribution in a homogeneous electrical field and in the case of constant relaxation time. Ohms' law and Joule's law can be immediately obtained from the solution.

Author's summary.

Erma, Victor A. Zur Thomas-Fermischen Gleichung bei hohen Temperaturen. Ann. Physik (6) **20** (1957), 345–348.

The equation of the title is

$$\beta''(s)=s\int_0^\infty x^{\frac{1}{2}}[\exp(x-\beta/s)+1]^{-1}dx.$$

The author is interested in evaluating the solution for small positive s from given initial values $\beta(0)=\alpha>0$, $\beta'(0)=a_2$. Since numerical integration is inconvenient, the author applies two iterations of Picard's method. Error estimates are given for the approximations so obtained. *Walter Gautschi* (Washington, D.C.).

Pathria, R. K. On the (relativistic) statistical thermodynamics of an assembly in mass-motion. Proc. Nat. Inst. Sci. India. Part A. **23** (1957), 168–177.

Statistical thermodynamics of an ideal relativistic gaseous assembly in mass-motion is studied by introducing the constraint of a fixed non-zero momentum. The transformation equations, connecting the observations in the laboratory system K and in the rest system K^0, are obtained for various thermodynamical quantities, and the invariance of the degree of degeneracy is brought out. The dynamical aspect of the results is discussed, showing thereby that the assembly behaves as if it possessed an inertial mass given by $(E+PV)/c^2$. (Author's summary.) *J. Ross* (Providence, R.I.).

Sundheim, Benson R. Transport processes in multicomponent liquids. J. Chem. Phys. **27** (1957), 791–795.

The partial differential equations for problems involving simultaneous transport of matter, heat, and electricity are shown to be separable if suitable linear combinations of the dependent variables are introduced. The treatment holds only if the transport coefficients involved are constant. *S. Prager* (Minneapolis, Minn.).

Bearman, Richard J.; and Kirkwood, John G. Statistical mechanics of transport processes. XI. Equations of transport in multicomponent systems. J. Chem. Phys. **28** (1958), 136–145.

The results of Irving and Kirkwood [same J. **18** (1950), 817–829; MR **12**, 230] are generalized to multicomponent mixtures. Diffusion coefficients, viscosities, and heat conductivities are obtained in terms of integrals involving the intermolecular potentials and non-equilibrium pair distribution functions. *S. Prager.*

See also: Structure of Matter: Prigogine. **Astronomy:** Erickson.

Elasticity, Plasticity

Onicescu, O. Variationsprinzipien, welche die inneren Verbindungen eines kontinuierlichen Mediums definieren. Rev. Méc. Appl. **1** (1956), no. 2, 7–13.

The author states variational equations for: (1) the motion of a continuous medium; (2) the effects of compressibility, pressure, and viscosity. His idea is to represent all of these phenomena by variational equations of a special type. Due to the author's use of a very compressed notation, the reviewer is unable to follow the argument. *N. Coburn* (Ann Arbor, Mich.).

Koiter, W. T. An elementary solution of two stress concentration problems in the neighborhood of a hole. Quart. Appl. Math. **15** (1957), 303–308.

Elementary solutions of the stress concentration problems for a strip under tension with a large circular hole, and a cylindrical bar with a large spherical hole, are determined by using beam theory and shell theory respectively. The stress concentration factor is found in both cases in the limit when $\lambda=1$, where λ is the ratio of the hole radius to the strip or bar radius. Previous solutions of these problems have been restricted to hole diameters which do not exceed half the strip or bar width. An alternative approach to the problem is also suggested. *R. M. Morris* (Cardiff).

Sokołowski, Marek. Some plane problems with boundary conditions in terms of displacements. Arch. Mech. Stos. **9** (1957), 439–454. (Polish and Russian summaries)

This paper deals with several plane stress or plane strain problems in the case of rigidly fixed boundaries in which the components of displacement (u, v) are both assumed to be zero. These problems are solved for an orthotropic material, assuming the loads to be in the form of concentrated forces acting within the region under consideration. Airy's stress function and Galerkin's displacement functions are used and solutions are obtained in closed forms, in the form of infinite systems of algebraic equations, or by means of integral equations. *R. M. Morris.*

Kalandiya, A. I. On contact problems of the theory of elasticity. Prikl. Mat. Meh. **21** (1957), 389–398.

A complex variable treatment of the following plane stress problem: — a concentrated load applied at the centre of a circular region, part of the boundary stress-free, the remainder free from tangential stresses but with prescribed normal displacements. The problem is reduced to solving an integro-differential equation of form occurring in the theory of wings of finite span. H. Multhopp's method [Luftfahrtforschung **15** (1938), 153–169] is used to obtain numerical results for some examples. *R. C. T. Smith* (Armidale).

Dovnorovič, V. I. A space contact problem concerning a hard punch with a surface of rotation represented by a polynomial in Cartesian coordinates. Prikl. Mat. Meh. **21** (1957), 272–278. (Russian)

A continuation of the work of L. A. Galin [same Prikl. Mat. Meh., **11** (1947), 281–284; MR **9**, 122] and others.

Rogoziński, Marian. An attempt to establish the theoretical foundations of the Moiré method of strain and stress analysis. Arch. Mech. Stos. **9** (1957), 191–210. (Polish and Russian summaries)

Stoppelli, Francesco. Su un sistema di equazioni integro-differenziali interessante l'elastostatica. Ricerche Mat. **6** (1957), 11–26.

The author continues his program of establishing the existence and uniqueness of solutions of the equations of finite deformation when the loads are sufficiently small. His earlier work [Richerche Mat. **3** (1954), 247–267; **4** (1955), 58–73; MR **17**, 554, 801] leaves unsolved only the case in which the loads have an axis of equilibrium. Here he considers a more general system containing the moment of an arbitrary vector. All solutions of the equations of elasticity are also solutions of this more general system, whose properties he determines. Interpretation in an elastic context is left for a later work. *C. Truesdell.*

Teodorescu, P. P. On the plane problem of the elastodynamics. Rev. Méc. Appl. **1** (1956), no. 2, 179–184.

For the dynamical equations of linear elasticity for isotropic materials, the author gives a solution in terms

of stress functions for plane strain and plane stress. In the latter case, certain approximations are involved. These could have been eliminated by interpreting plane stress as generalized plane stress. Omitted detail in the derivation leaves doubt in the reviewer's mind as to the completeness of this solution. *J. L. Ericksen.*

Filonenko-Borodič, M. M. On Lamé's problem for a parallelepiped in the general case of surface loads. Prikl. Mat. Meh. **21** (1957), 550–559. (Russian)

The author considers the equilibrium of a rectangular parallelepiped loaded on all six faces by a given stress distribution whose shear components satisfy the reciprocal law (e.g. $Y_z=Z_y$) at each edge. The given loading stresses define a matrix

$$M=\begin{Vmatrix} \xi_x & \xi_y & \xi_z \\ \eta_x & \eta_y & \eta_z \\ \zeta_x & \zeta_y & \zeta_z \end{Vmatrix}$$

whose elements reduce to the given loading stresses on the faces

$$x=0, a;\ y=0, b;\ z=0, c$$

but are otherwise arbitrary. It is then shown how this arbitrariness can be modified by the addition of a matrix M_0 such that the elements of M_0 are zero on the faces, the matrix $M+M_0$ is symmetric and its elements satisfy the differential equations of equilibrium.

L. M. Milne-Thomson (Providence, R.I.).

Lašmanova, I. A.; and Novožilov, V. V. Torsion of tubes under constraint. Leningrad. Gos. Univ. Uč. Zap. **217.** Ser. Mat. Nauk **31** (1957), 254–271. (Russian)

This paper is devoted to the problem of torsion of cylindrical tubes with constant thickness where the ends are constrained to remain plane. The problem is solved using Vlasov's theory of cylindrical membrane-shells modified by Novožilov's treatment [Theory of thin shells, Gos. Izdat. Sudostroit. Lit., Moscow, 1951; MR **17**, 915]. This procedure improves on the results which were obtained by other authors during the period from 1939 to 1949 (Umanski, Zvolinski, Dzanelidze).

Starting from the basic differential equation of cylindrical shells in complex form, the author reduces the problem to the determination of fundamental functions and characteristic numbers with boundary conditions. Then the solution of the governing differential equation reduces to a nonhomogeneous nonlinear integral equation. The general formulae for forces and displacements are deduced, and the role of boundary conditions is treated. At the end of the paper, the author discusses the accuracy of the results. The conclusion is that, since the error in the case of a very thin tube, which is useful in technical practice, is only 1 to 2 per cent, the significance of the results of this paper are only of an academic rather than technical character. [Cf. Netrebko, Vestnik Moskovsk. Univ. **11** (1956), no. 6, 11–25; MR **18**, 613.]

D. P. Rašković (Belgrade).

Chandra Das, Sisir. On the stresses in a composite truncated cone due to shearing stresses on the curved surface. Indian J. Theoret. Phys. **4** (1956), 89–92.

The following composite beam torsion problem is solved: In spherical coordinates (r, θ, φ) the member is assumed to be bounded by the spherical surfaces $r=a$ and $r=b$ $(b>a)$ and by the conical surface $\theta=\alpha$. The coaxial surface $\theta=\beta$ is taken as the surface of separation of two different elastic materials. The surfaces $r=a$ and $r=b$ are assumed fixed and the outer surface $\theta=\alpha$ is subjected to a uniformly distributed shearing stress. *L. E. Payne.*

Verma, G. R. Stresses in a circular cylinder and in a paraboloid of revolution due to shearing forces produced by circular rings of the curved surface. Indian J. Theoret. Phys. **4** (1956), 93–98.

The author solves the following two torsion problems. (1) A finite isotropic circular cylinder with plane ends free of traction twisted by two equal and opposite torque rings applied to the lateral surface; (2) an isotropic cylinder formed by two intersecting confocal paraboloids. The portion of the surface formed by one of the paraboloids is fixed while a torque ring is applied to the other portion of the surface. *L. E. Payne* (College Park, Md.).

Strelbitskaya, A. I. Centre of flexure of a thin-walled profile beyond the elastic limit. Akad. Nauk Ukraïn. RSR. Prikl. Meh. **3** (1957), 295–305. (Ukrainian. Russian and English summaries)

In this paper the limiting state of a thin-walled section having one axis of symmetry is considered, and formulas are proposed for determining the position of the centre of flexure in those sections of a bar where shearing stresses are absent or negligible. *From the author's summary.*

Ugodčikov, A. G. Torsion of hollow prismatic rods. Akad. Nauk Ukrain. RSR. Prikl. Meh. **2** (1956), 217–223. (Ukrainian. Russian summary)

Ling, Chih-Bing. Stresses in a circular cylinder having a spherical cavity under tension. Quart. Appl. Math. **13** (1956), 381–391.

The stress system under consideration is satisfied by a biharmonic stress function; the construction of the biharmonic function satisfying the boundary conditions proceeds, by linear superposition, as follows: to the function appropriate to the problem without circular cavity are added two sets of biharmonic functions, each of which gives no traction on the surface of the cylinder and at the same time gives no stress at the infinitely far distant ends of the cylinder. The boundary conditions at the surface of the cavity are satisfied by adjusting the coefficients of superposition. The sets of biharmonic functions are derived by differentiation from two biharmonic functions, each of which has a singularity at the origin.

W. F. Freiberger (Providence, R.I.).

Langefors, B. Algebraic methods for the numerical analysis of built-up systems. Svenska Aeroplan A. B. Tech. Note No. **38** (1957), 55 pp.

Elastic structures may be studied by considering them as topological structures built up by one-, two-, and three-dimensional building-blocks. The superimposed algebraic structure has, however, the disconcerting property of presenting singular relations between stresses and strains. As a result, the accompanying influence matrices have varying degrees of degeneracy. The author surmounts these difficulties — at least for pin-jointed structures — by ignoring the underlying topological structure (with the exception of the vertices), and by developing, in their place, an abstract algebraic theory of "links". Of course, within the terminology of linear vector spaces the underlying topology still leaves its footprints. *G. Kron.*

Vorob'ev, L. N. On the determination of point displacements in strained systems. Dokl. Akad. Nauk SSSR (N.S.) **109** (1956), 465–468. (Russian)

In the theory of structures, as is well known, the equation of virtual work and Mohr's equation to determine the small displacements of strained systems, with the corresponding small strains being given, are widely utilized. The author generalizes these equations, to be applicable to the case of finite displacements comparable in magnitude to the considered structure itself. *T. P. Anđelić.*

Schumpich, G. Beitrag zur Kinetik und Statik ebener Stabwerke mit gekrümmten Stäben. Österreich. Ing.-Arch. **11** (1957), 194–225.

This paper develops transfer matrices for vibrations of curved beams for important elementary practical cases, viz.: elastic circular arc without mass, rigid circular arc (with mass), a discrete mass and an elastic joint. Most problems can be solved by the replacement of the actual structure with a number of these elements. The method is illustrated by three examples. *W. T. Koiter* (Delft).

Béres, Elek; Lovass-Nagy, Viktor; und Szabó, János. Über eine Anwendung der Hypermatrizen bei der Berechnung von räumlichen Fachwerken mit zyklischer Symmetrie. Magyar Tud. Akad. Mat. Kutató Int. Közl. **1** (1956), 559–576 (1957). (Hungarian. Russian and German summaries)

Bekanntlich lassen sich die linearen algebraischen Gleichungen, welche zur Berechnung von Spannkräften räumlicher Fachwerke dienen, zu einer einzigen Matrizengleichung zusammenfassen. Hat das Fachwerk eine zyklische Struktur, dann lässt sich die Koeffizientenmatrix dieser Matrizengleichung in zyklische Blöcke zerlegen. Mit Hilfe der Egerváry'schen Hypermatrizenalgorithmus entwickeln die Verfasser eine Methode zur Spektralzerlegung der Koeffizientenmatrix. Hierbei erscheint die aus zyklischen Blöcken bestehende Hypermatrix als eine Summe, deren Glieder direkte Produkte von Eigenwert-Matrizen und der entsprechenden Eigen-Dyaden sind. Diese Methode liefert einheitliche und übersichtliche Lösungsformeln zur Berechnung der Spannkräften statisch bestimmter wie statisch unbestimmter räumlicher Fachwerke, sie führt nämlich die Inversion der Koeffizientenmatrix zur Berechnung von Reciprokmatrizen der Eigenwert-Matrizen der Blöck zurück.

Zusammenfassung der Autoren.

Béres, Elek. Über eine Anwendung der Hypermatrizen zur Berechnung von räumlichen Stabwerken mit zyklischer Symmetrie. Magyar Tud. Akad. Mat. Kutató Int. Közl. **1** (1956), 577–592 (1957). (Hungarian. Russian and German summaries)

Es ist bekannt, dass die mathematische Untersuchung von Problemen der technischen Festigkeitslehre auf Grund des Hookeschen Gesetzes auf lineare Gleichungs Systeme führen. Hat ein räumliches Stabwerk eine Aufbau von zyklischer Symmetrie, dann zeigt auch die Koeffizienten-Matrix des Gleichungs-Systems eine zyklische Regelmässigkeit: d.h. sie ist in zyklische Blöcke zerlegbar. Es ist von mehreren Verfassern versucht, um diese zyklische Regelmässigkeit auszunützen (zum Beispiel: H. Reissner, L. Mann und W. Kaufmann), aber diese Verfasser benützen nicht die Möglichkeiten der Matrizenrechnung. Verfasser zeigt, dass falls die Koeffizienten-Matrix in zyklische Blöcke zerlegt ist, mit Hilfe der Hypermatrizen-Theorie einheitliche und übersichtliche Lösungsformeln zu gewinnen sind. Diese Lösungsformeln sind auch zur maschinellen Berechnung gut anwendbar.

Zusammenfassung des Autors.

Novožilov, V. V. On the center of shear. Prikl. Mat. Meh. **21** (1957), 281–284. (Russian)

Dol'berg, M. D. On a generalization of Bubnov's problem. Ukrain. Mat. Žurnal **3** (1951), 433–448. (Russian)

Hu, Hai-Chang. On the large deflection of a circular plate under the combined action of a uniformly distributed load and a concentrated load at the center. Sci. Sinica **4** (1955), 251–261.

The author discusses the problem described in the title, but references indicate that he is unaware of work by Way [Trans. A.S.M.E. **56** (1934), 627–636; and A. Nádai, Die elastische Platten, Springer, Berlin, 1925]. The plate considered has no radial stress at edge. Rotation at the edge is prevented. A solution is obtained in the form of a power series and resulting constants in the solution are presented. The author states, however, "We may conclude that the above formulas are suitable only for small values..., for large values··· they either converge too slowly... or are even divergent". The case in which the concentrated load at the center is oppositely directed to the distributed load and sufficient to prevent central deflection is discussed in more detail. None of the results are presented in the form of stress or deflection vs. load. *S. Levy.*

Bassali, W. A.; and Dawoud, R. H. Bending of a circular plate with an eccentric circular patch symmetrically loaded with respect to its centre. Proc. Cambridge Philos. Soc. **52** (1956), 584–598.

Bassali, W. A. Bending of an elastically restrained circular plate under a linearly varying load over an eccentric circle. Proc. Cambridge Philos. Soc. **52** (1956), 734–741.

For non-symmetrical loadings, solutions of bending of thin circular plates are given by several authors [A. Clebsch, Theorie der Elastizität fester Körper, Teubner, Leipzig, 1862; W. Flügge, Die strenge Berechnung von Kreisplatten unter Einzellasten, Springer, Berlin, 1928; Gran Olsson, Ing.-Arch. **9** (1938), 108–115; Dawoud, Proc. Math. Phys. Soc. Egypt. (in the Press)] for the linearly varying load $p=p_0 \cos \varphi$ over the whole plate or over a concentric circle, under general boundary conditions defining certain types of constraints at the boundary [W. A. Bassali and R. H. Dawoud, Proc. Cambridge Philos. Soc. **52** (1956), 584–598; see the article listed above]. In this last paper the complex variables method was applied to obtain solutions for a circular plate with an eccentric circular patch symmetrically loaded with respect to its centre and subjected to the general boundary conditions introduced.

In the present paper the author presents a solution for a thin circular plate under the same boundary conditions, which is acted upon by the linearly varying load $p=p_0 r \cos \varphi$ or, more generally, by the load $p=p_0 r^n \cos n\varphi$ (n being an integer) over an eccentric circle, where r and φ are measured from the centre of the circle and the common diameter of the plate and the circle. The problem affords an example of the usefulness and simplicity of the complex potential method. *R. Gran Olsson.*

Bassali, W. A. Transverse bending of a thin circular plate loaded normally over an eccentric circle. Proc. Cambridge Philos. Soc. **52** (1956), 742–749.

The complex variable method was applied by W. A. Bassali and R. H. Dawoud [Proc. Cambridge Philos. Soc. **52** (1956), 584–598; see the article listed second above] to obtain solutions for a circular plate having an eccentric circular patch symmetrically loaded with respect to its centre, under the general boundary conditions defining certain types of constraints at the boundary. The author also found the solution for a linearly varying load over an eccentric under the same boundary condition [see the paper reviewed above]. In the present paper the power of the complex variables method is exhibited by finding the appropriate complex potentials corresponding to the load $p=p_0r^m\cos n\varphi$ $(n>1)$ over an eccentric circular patch, where r and φ are measured from the centre of the patch and the common diameter of the plate and the patch. Since the two cases $n=0$ and $n=1$ require special consideration in the two papers mentioned above, this paper completes the solution of the problem of a circular plate with an eccentric circular patch symmetrically loaded with respect to the common diameter of the plate and the patch, where the load, in this case, is expressible in the form $p=\sum_{m,n}p_{m,n}r^m\cos n\varphi$. For a clamped boundary the solution is obtained in finite terms. In this paper there are some small misprints.

R. Gran Olsson (Trondheim).

Bassali, W. A. Transverse bending of infinite and semi-infinite thin elastic plates. I. Proc. Cambridge Philos. Soc. **53** (1957), 248–255.

In the present paper the complex variable method is applied to find the appropriate complex potentials and deflexion for an infinite thin plate with an inner circular boundary subject to boundary conditions defining certain types of boundary constraint, including those of rigidly clamped and hinged boundaries introduced by W. A. Bassali and R. H. Dawoud [Proc. Cambridge Philos. Soc. **52** (1956), 584–528; see the article listed third above]. The outer edge of the plate is assumed to be free, and the plate itself is loaded over a circular patch, the load being symmetrical with respect to the centre of this patch. The limiting case in which the circular boundary degenerates to a straight line is also considered. The solutions given are based on the classical theory of bending of thin plates and are exact within the assumptions underlying this theory.

R. Gran Olsson (Trondheim).

Bassali, W. A. Thin circular plates supported at several points along the boundary. Proc. Cambridge Philos. Soc. **53** (1957), 525–535.

The bending of a thin circular plate is considered for a number of cases where the plate is supported only at discrete points along the boundary. The support points on the boundary are assumed to be located symmetrically with respect to some diameter of the boundary circle. Solutions are found for plates subjected to a lateral pressure which is applied either around the circumference of, or distributed over the area of, a smaller circle located eccentrically inside the plate boundary. The center of this loaded circular patch is assumed to be located on the diameter of symmetry of the supports. The notation and methods used in the development are essentially the same as are given in Section 51, pp. 270–273 of S. Timoshenko, "Theory of plates and shells" [McGraw-Hill, New York, 1940], for dealing with circular plates which are supported at several points along the boundary. In previous work by the author [Proc. Camb. Phil. Soc., **52** (1956), 734–741; see the article reviewed third above] and with R. H. Dawoud [Proc. Camb. Phil. Soc., **52** (1956), 584–598 see the article listed fourth above]), solutions were developed for the circular plate with simply supported boundary all around and under the same loadings considered in the present work. These solutions were given in the form of functions w_1 and w_2 which are the deflections in region 1, inside the eccentric circle, and in region 2, outside the eccentric circle, respectively. The present work deals with the determination of a function w_0, which satisfies $D\nabla^4w_0=0$ in both regions 1 and 2, and whose derivatives take on such values at the plate boundary as to make w_0+w_2 satisfy the assumed conditions of pointwise support. In this paper w_0 and the complete solutions w_0+w_1 and w_0+w_2 are developed, or are capable of being determined from given coefficients, for a number of special cases of loading and support, including loading by a concentrated force applied on the diameter of symmetry or by a concentrated couple applied with axis perpendicular to this line. As a particular example, the author develops a formula for the central deflection of a uniformly loaded circular plate with three equispaced supports along the boundary; his result differs, however, from that given for this same problem in Timoshenko, loc. cit., bottom of page 271. *W. Nachbar.*

Ku, Chiu-Lin. On the large deflection of elastic circular membrane with initial tension under uniformly distributed load. Sci. Sinica **5** (1956), 423–443.

This paper discusses the problem of large deflection of an elastic circular membrane with initial tension, uniformly distributed load, and a clamped edge. Results are presented in the form of non-dimensional stress and deflection for non-dimensional load. A table is given for a range of values of Poisson's ratio and of a parameter in the analysis. The results show linear ranges are present, for both very small and very large deflections. The dynamic error of a membrane apparatus of particular interest to the author is discussed. *S. Levy.*

Cooper, R. M. Cylindrical shells under line load. J. Appl. Mech. **24** (1957), 553–558.

The author uses a system of equations which are a special case of the shallow-shell equations of Naghdi [Quart. Appl. Math. **14** (1956), 331–333; MR **19**, 339] in which only the transverse-shear deformation terms are retained. The general solution ot the system is found for a simply supported cylindrical shell and the coefficients which result are obtained for a line load sinusoidally distributed along a generator. The case of uniform load on a segment of a generator is then obtained by superposition.

Use of a numerical example shows that the stresses are in excellent agreement with those obtained from the Donnell equations. However, the Donnell equations predict a radial displacement which is in error by 20 per cent for the shell geometry considered.

H. D. Conway (Ithaca, N.Y.).

Behlendorff, Erika. Über Randwertprobleme bei Häuten und dünnen Schalen im Membranspannungszustand. Z. Angew. Math. Mech. **36** (1956), 399–413. (English, French and Russian summaries)

This paper discusses the equilibrium of infinitely flexible membranes of positive curvature under given surface loads and specified tangential stresses at the boundary. By using exterior differential forms, a first

order system of equations of equilibrium is derived and it is shown that the problem, if solvable, is statically determined. In order to obtain conditions for solvability an adjoint system is introduced. This system coincides with that of the inextensional infinitesimal displacements of the membrane and thus it is found that a necessary and sufficient condition for equilibrium is that the surface load plus the stresses at the boundary be orthogonal to all inextensional displacements of the membrane for which the displacement vector at points of the boundary projects on the tangent plane along a vector tangent to the boundary. This is equivalent to three integral conditions on the loads and boundary stresses. These results are then applied to the case of spherical cap.

A. P. Calderón (Cambridge, Mass.).

Iškova, A. G. Some generalizations concerning solutions of problems of bending of a round plate and an infinite rod supported by an elastic half-space. Prikl. Mat. Meh. **21** (1957), 287–290. (Russian)

A continuation of the article reviewed in MR **9**, 122.

Danilyuk, I. I. On integral representations of solutions of certain elliptical systems of the first order upon surfaces and their use in the theory of thin shells. Dokl. Akad. Nauk SSSR (N.S.) **109** (1956), 17–20. (Russian)

Livšic, Ya. D. Bending of plates with fixed contour. Akad. Nauk Ukrain. RSR. Prikl. Meh. **2** (1956), 51–66. (Ukrainian. Russian summary)

Herrmann, George; and Mirsky, I. Three-dimensional and shell-theory analysis of axially symmetric motions of cylinders. J. Appl. Mech. **23** (1956), 563–568.

Stadelmaier, Hans H. Spannungsfeld der auf den Rand einer elastisch anisotropen Halbebene wirkenden Tangentialkraft. Z. Angew. Math. Phys. **8** (1957), 285–290.

The stress distribution obtained by solving the two-dimensional problem in an anisotropic medium, with boundary conditions of a concentrated tangential load acting on the boundary of a semi-infinite plate, is purely radial. The solution is given in closed form and is combined with the solution for a concentrated normal load to solve the problem of an inclined force acting on the boundary. *Author's summary.*

★**Болотин, В. В. [Bolotin, V. V.] Динамическая устойчивость упругих систем. [Dynamic stability of elastic systems.]** Gosudarstv. Izdat. Tehn.-Teor. Lit., Moscow, 1956. 600 pp. 16.50 rubles.

This book presents an attempt at a systematic exposition of the general theory of dynamic stability of elastic systems, and includes many applications. The book is based on the author's own investigations on this problem, published in Russian in different scientific periodicals and journals.

The book is divided into three parts. The first part is devoted to the simplest problems of dynamic stability which do not require a complicated mathematical apparatus. The second part gives first a mathematical introduction to the matrix calculus and to the theory of integral equations. Then the properties of the general equations of dynamic stability are thoroughly analyzed; the methods for the determination of boundaries of domains of instability are studied; and, finally, the amplitudes of the parametric vibrations are investigated. The third part deals only with technical applications. The examples are chosen to serve, on the one hand, as illustrations of the quoted general methods and, on the other, to demonstrate the solution of concrete practical problems. *T. P. Anđelić* (Belgrade).

Reissner, Eric. On axi-symmetrical vibrations of shallow spherical shells. Quart. Appl. Math. **13** (1955), 279–290.

In a previous paper [J. Appl. Phys. **17**, 1038–1042 (1946), MR **8**, 361] the author had given the solution of the differential equations for axi-symmetrical vibrations of shallow spherical shells in terms of Bessel functions. The present work introduces an approximation, based on the fact that for transverse vibrations of shallow shells the magnitude of the longitudinal inertia terms is negligibly small compared with the magnitude of the transverse inertia terms [E. Reissner, Q. Appl. Math. **13**, 169–176 (1955); MR **16**, 1070]. It is thus possible to solve the differential equations numerically, which is here done for three specific problems: 1. Determination of the lowest frequency of free vibrations for a shell segment with clamped edge. 2. Determination of the frequencies of free vibrations of a shell segment with free edge. 3. Forced vibrations due to point load at apex of the shell.

W. F. Freiberger (Providence, R.I.).

★**Bishop, R. E. D.; and Johnson, D. C. Vibration analysis tables.** Cambridge University Press, New York, 1956. viii+59 pp. $2.00.

This is a collection of formulae and tables, many of which have appeared separately in various engineering journals, which is extracted from a forthcoming book by the same authors [Mechanics of Vibration, Cambridge University Press]. They are related to the various modes of vibration of strings, shafts, bars and beams with various end-conditions.

A series of tables, computed by Joy Elliott, gives $\sin x \sinh x$, $\cos x \cosh x$, $\cos x \cosh x \pm 1$, $\cos x \sinh x \pm \sin x \cosh x$, $\sin x \pm \sinh x$, $\cos x \pm \cosh x$ for $x=0(.05)11$, to about 5S [cf. R. E. D. Bishop, Proc. Inst. Mech. Engrs. **169** (1955), 1031–1050; 170, 1956; MR **18**, 614].

The second series of tables, mainly due to Dana Young and R. P. Felgar [Univ. of Texas, Bur. of Engrg. Res., Engrg. Res. Bull. **4913** (1949)] gives the (first five) characteristic functions (together with their first three derivatives) of the differential equations for the clamped-clamped (and free-free), clamped-free, and clamped-pinned (and pinned-clamped) cases. Thus we have the first five positive zeros of $\cos x \cosh x \pm 1$ and $\tan x - \tanh x$, given to 6S, together with various related quantities. For instance, in the last case, $\varphi_r(x)=\cosh xx_r - \cos x_r x - \sigma_r(\sinh xx_r - \sin xx_r)$, and its derivatives, are given for $x=0(.02)1$, to 5D, where $\sigma_r=\cot x_r=\coth x_r$ and $\tan x_r=\tanh x_r$.

There is a short bibliography which does not mention K. Hohenemser and W. Prager, Dynamik der Stabwerke. Eine Schwingungslehre für Bauingenieure [Springer, Berlin, 1933] whose contents have been described by W. Prager [Math. Tables Aids Comput. **1** (1943), 101–103].

The format and printing are excellent; we have not checked the tables. *John Todd* (Pasadena, Calif.).

Kynch, G. J.; and Green, W. A. Vibrations of beams. I. Longitudinal modes. Quart. J. Mech. Appl. Math. **10** (1957), 63–73.

The authors attempt to calculate dispersion curves of

longitudinal stress waves in uniform bars of certain non-circular sections by means of a perturbation technique. The boundary curve is expressed in polar coordinates in the form $r=a(1+\sum b_s \cos s\theta)$, the coefficients b_s being small. The general solution is written as a linear combination of the known solutions for a circular cylinder. The boundary conditions expressed as Fourier series require the vanishing of two determinants, one of which includes a solution corresponding to longitudinal vibrations, the other containing a solution for torsional oscillations. The elements are expanded in powers of b_s and transformations are carried out in order to furnish values of phase velocity as functions of wavelength correct to terms of order b_s^2. It was not found possible to estimate errors due to the neglect of third order terms. Comparisons of results for a square and a rectangle with experimental results of Morse indicate that the perturbation method may be in error by about 3 per cent for the square; for the rectangle there is a larger discrepancy at one frequency which is probably due to a coupling of two degenerate modes of the circular cylinder, which causes the perturbation theory to fail. *P. S. Symonds* (Swansea).

Green, W. A. Vibrations of beams. II. Torsional modes. Quart. J. Mech. Appl. Math. **10** (1957), 74–78.

The method of the paper reviewed above is applied to obtain a second-order perturbation theory for the torsional oscillations of uniform bars of non-circular section. Phase velocities are calculated for a square, rectangular and elliptic section. Results are subject to uncertainty for the reasons discussed in the preceding review for the longitudinal modes. *P. S. Symonds* (Swansea).

Bishop, R. E. D. The vibration of frames. Proc. Inst. Mech. Engrs. **170** (1956), 955–967, discussion 968.

The paper is a sequel to Bishop, same Proc. **169** (1955), 1031–1050 [MR **18**, 614]. This previous work gave tables for functions required in the working of the receptance method of analysis. The present paper gives examples of simple frames, calculated manually rather than by electronic machine, and determines resonant frequencies and modes. These are compared with experimental results on model frames; the agreement is almost perfect.

J. Heyman (Providence, R.I.).

Kempner, Joseph. Stability equations for conical shells. J. Aero. Sci. **25** (1958), 137–138.

The author rewrites V. S. Vlasov's equations for (linear) stability theory of shells [Prikl. Mat. Meh. **8** (1944), 109–140; MR **7**, 42; **13**, 300] in terms of equations for displacements. The result is a single eighth-order equation for normal deflection, and two fourth-order equations relating the displacement components in the shell middle surface to the normal displacement. *W. T. Koiter.*

Robinson, A. Wave propagation in a heterogeneous elastic medium. J. Math. Phys. **36** (1957), 210–222.

The method of characteristics is used to trace the motion, through a material of slowly varying elastic properties, of a wave involving discontinuity in material velocity. *J. W. Craggs* (Newcastle-on-Tyne).

Robinson, A. Transient stresses in beams of variable characteristics. Quart. J. Mech. Appl. Math. **10** (1957), 148–159.

The equations due to Timoshenko of transverse motion of a beam are studied for the case of stiffness and inertia coefficients (appropriate to both shear and bending deformations) assumed to be functions of distance along the beam. The two families of characteristic curves are defined by the same relations as for the uniform beam, namely: $dx/dt=\pm C_\sigma$, $dx/dt=\pm C_\beta$, where C_σ, C_β are velocities of propagation in shear and bending, respectively. The variation in discontinuity in shear force across the front of a shear wave, and that in bending moment across the front of a bending wave, are deduced. In the particular case of a beam initially at rest and subjected to a concentrated impulse, it is found that there is a jump in bending moment as well as in shear force across the shear wave front; the magnitudes of these jumps are calculated, making use of results previously obtained by R. P. N. Jones [same Quart. **8** (1955), 373–384] for a uniform beam. *P. S. Symonds* (Swansea).

Musgrave, M. J. P. On whether elastic wave surfaces possess cuspidal edges. Proc. Cambridge Philos. Soc. **53** (1957), 897–906.

In some cases, elastic wave surfaces for anisotropic media have cuspidal edges. The author derives inequalities which must be satisfied by the elastic constants for such edges to exist. *J. L. Ericksen* (Baltimore, Md.).

Huan, Tun. Slow elastic waves. Prikl. Mat. Meh. **21** (1957), 381–388. (Russian)

This paper is concerned with the propagation of surface elastic waves along the plane interface between a semi-infinite elastic solid and a layer of heavy liquid. It is shown that harmonic waves with velocity from zero up to approximately $\sqrt{(gH)}$ can be propagated along the surface, where H is the depth of the liquid layer, and g the acceleration due to gravity. Because this speed is very much less than that of all previously discussed elastic waves the author calls them "slow elastic waves". A formula is derived relating the amplitude of such a slow wave to the amplitude of the wave it produces at the free liquid surface. The dispersion of the waves is also discussed. The theory is illustrated by numerical examples.

I. N. Sneddon (Glasgow).

Katasonov, A. M. Propagation of spherical thermal visco-elastic perturbations. Vestnik Moskov. Univ. Ser. Mat. Meh. Astr. Fiz. Him. **12** (1957), no. 3, 39–49. (Russian)

★**Prager, W. Non-isothermal plastic deformation.** Tech. Rep. No. **4**, Nonr 5–562(20), Division of Engineering, Brown University, Providence, R. I., 1957. 12 pp.

A workhardening rigid-plastic solid is considered such that the strain-rate depends linearly and homogeneously on the rates of change of stress and temperature. Time-dependent creep is thereby excluded from the theory. When the temperature-rate vanishes, the generally accepted isothermal equations are recovered, with identical yield and plastic potential functions depending on strain-history. Some special constitutive laws are suggested in which yield depends only on the final (infinitesimal) strain and not on the preceding strain path.

When changes in geometry can be neglected, the following generalization of the isothermal uniqueness theorem is proved: in a body in a given state of stress and strain, there can not be more than one continuous velocity

field corresponding to prescribed temperature-rate through the volume and velocity or traction-rate on the surface. *R. Hill* (Nottingham).

Bland, D. R. The associated flow rule of plasticity. J. Mech. Phys. Solids **6** (1957), 71–78.

The author states in his summary, "The workhardening and linearity hypotheses introduced by Drucker are logically equivalent to the hypotheses of the existence of the plastic potential and of its identity with the yield function. Since neither of these sets of hypotheses is deducible from established physical laws, the author suggests that they are alternative statements of a new physical law". It is true that my mechanical-thermodynamic definition of stable work-hardening and perfectly plastic time-independent materials leads to convexity of all yield surfaces and generalized normality of the plastic strain increments. Also conversely, convexity and normality require that when an external agency adds forces to an already loaded elastic-plastic body the work done by the external agency is positive. However, the two approaches are not fully equivalent and as a result, the author is led to some reasonable but still added regularity conditions at singular points of the yield surface. The mechanical-thermodynamic postulate has a twofold advantage. It is predictive, as for example in connection with slip theory of Batdorf and Budiansky. If overall plastic deformation is computed from simple slip on some or all planes, and if such slip follows a hardening law, the resulting yield surface for a single crystal or a polycrystalline metal must be convex. The second advantage is that the postulate is essentially in the language of irreversible thermodynamics and is suggestive for time-dependent materials. Nevertheless, the (flattering) conclusion of a new physical law requires qualification. Unstable materials, such as mild steel at the upper yield point, do exist and true time-independence does not. My statement about positive work by the external agency merely defines a very extensive class of stable materials. Only in this narrow sense can it be considered a law rather than a postulate or definition. *D. C. Drucker.*

Boyce, W. E.; and Prager, W. On rigid workhardening solids with singular yield conditions. J. Mech. Phys. Solids **6** (1957), 9–12.

Hill's [same J. **4** (1956), 247–255; MR **18**, 83] results for a rigid workhardening solid with a regular yield surface are extended to include a singular yield surface. Uniqueness is first proved for the following case: given the complete strain history of every particle, the surface tractions on part of the boundary S_f, the velocities on the remainder of the boundary S_v, and no body forces, then the strain rates are unique throughout the body. If $S_v \neq 0$, this implies unique velocities, but the stress rates are not necessarily unique. The paper concludes by showing that an appropriate generalization of Hill's extremum principles is valid for the case of singular yield conditions. *P. G. Hodge, Jr.* (Chicago, Ill.).

Dodeja, L. C.; and Johnson, W. On the multiple hole extrusion of sheets of equal thickness. J. Mech. Phys. Solids **5** (1957), 267–280.

Plane strain velocity fields are constructed for the extrusion of a sheet through a square die with one, two, or three orifices. The two cases of perfectly smooth and perfectly rough container walls are considered. The theoretical results are compared with experimental ones for pure lead, tellurium lead, and pure tin. In general, the experimental pressures are higher than predicted, although the pressure predicted from a velocity field should be an upper bound. The authors suggest that this discrepancy may be due to flashing at high pressures. *P. G. Hodge, Jr.*

Shepherd, W. M.; and Gaydon, F. A. Plastic bending of a ring sector by end couples. J. Mech. Phys. Solids **5** (1957), 296–301.

A circular ring sector is subjected to forces equivalent to a pure couple on its plane faces; the curved faces remain stress free. The magnitude of the couple required to produce a fully plastic state of stress in the ring sector is determined. The ring sector is assumed to be in a state of plane stress. Solutions as function of the ring radii are obtained for the two cases corresponding to the von Mises and Tresca yield criteria. In each case, stress and velocity fields are obtained which are both statically and kinematically admissible. *P. G. Hodge, Jr.* (Chicago, Ill.).

Rosenblyum, V. I. The time to destruction of a rotating disc in conditions of creep. Prikl. Mat. Meh. **21** (1957), 440–444. (Russian)

"A process of creep in metal components at high temperatures, during which there is the passage of a shorter or longer period of time, usually results in break-up.

"In connection with this, the important problem of determination of the time to destruction (period of service) of the component arises. Normally, when creep problems occur in the case of small strains, this problem is not discussed. Here, two types of break-up may be distinguished; i.e., brittle, or without deformation, and viscous, which is a result of large creep deformations.

"The simple approximate solution dealt with here, for the case of large strains in a rotating disc, discloses that the deformations begin to grow rapidly only after the passage of a certain finite length ot time t_*, where $t=t_*$ can approach infinity.

"Such types of solution allow the use of t_* as a measure of the time of service of the disc which is undergoing creep (and indeed of the conditions under which its break-up occurs in the second manner).

"A similar approach has been used recently by N. J. Hoff [J. Appl. Mech. **20** (1953), 105–108] in the problem of stretching a cylindrical rod; the solution obtained by Hoff is supported by the experimental data in the paper quoted". (From author's summary.) *H. G. Hopkins.*

Mikeladze, M. Š. A study of rotating anisotropic plates in a rigid-plastic and elastic-plastic state. Akad. Nauk Ukraïn. RSR. Prikl. Meh. **3** (1957), 260–268. (Ukrainian. Russian and English summaries)

Savin, G. N. Development of investigations on the theory of elasticity, applied mechanics and strength in the Ukraine during the 40 years of Soviet government. Akad. Nauk Ukraïn. RSR. Prikl. Meh. **3** (1957), 241–259. (Ukrainian)

Truesdell, Clifford. Sulle basi della termomeccanica. I, II. Atti Accad. Naz. Lincei. Rend. Cl. Sci. Fis. Mat. Nat. (8) **22** (1957), 33–38, 158–166.

These two notes are concerned with the mechanical and thermodynamical properties of a heterogenous continuous medium within which the different homogeneous constituents are diffusing. The author derives the conditions

that must be satisfied by the physical quantities associated with each of the constituents so that the laws of conservation of mass, momentum and energy are satisfied in the medium and finds the conditions under which an equation of state for the medium exists. He points out an analogy with the Maxwell-Boltzmann theory of gases.

D. R. Bland (Manchester).

Sternberg, Eli. Transient thermal stresses in an infinite medium with a spherical cavity. Nederl. Akad. Wetensch. Proc. Ser. B. **60** (1957), 396–408.

The author considers the transient thermoelastic response of an infinite medium to a sudden uniform change in the temperature of the surface of a spherical cavity. It is assumed that the distribution of temperature throughout the solid is governed by the classical equation of the conduction of heat, omitting the term introduced by Biot [J. Appl. Phys. **27** (1956), 240–253; MR **17**, 1035], and that the distribution of stress is determined by the equations of equilibrium (not motion) and the Duhamel-Neumann stress-strain relation. The solution therefore refers to a quasi-static change in the solid; the quantitative influence of the Biot term and of the inertial effects is not likely to be appreciable, and the solution derived here is applicable except when the time rate of change of the temperature applied to the cavity wall is high. This solution, which is derived with the use of Laplace transforms in the time variable, is obtained in a closed form involving error functions. Numerical values are given for the stress and temperature fields for the case of a constant temperature acting on the boundary of the cavity. The time and space variations of the temperature and of the stress field are discussed in detail and illustrated graphically.

I. N. Sneddon (Glasgow).

See also: Classical Thermodynamics, Heat Transfer: Li and Ting.

Structure of Matter

★ **Prigogine, I. The molecular theory of solutions. With the collaboration of A. Bellemans and V. Mathot.** North-Holland Publishing Company, Amsterdam; Interscience Publishers, Inc., New York, 1957. xx+448 pp. $13.25.

Many workers in the expanding field of chemical physics will be very grateful to Professor Prigogine and his associates for the care with which they have written this monograph. Experimenters will find a competent theoretical discussion of most of the topics which the title suggests, in which even the mathematics is simple and transparent. Theorists will find a very comprehensive account of the properties of solutions which is up-to-date in content (as far as possible in a book on a rapidly developing subject) and method. There is no discussion of the difficult topic of aqueous solutions and electrolytes, or of surface phenomena, which have been treated separately [I. Prigogine and R. Defay, Traité de thermodynamique···, t. 3, Desoer, Liège, 1951]. But there is no other important omission.

It would be easy to criticize this book on the grounds that the physical models and mathematical treatment are considerably over-simplified. This is the fault not of Prigogine but of the subject at its present stage of development. Indeed, the authors have been careful to point out the defects of the models and the ways to look for improvement. It is likely that when the more exact methods are available they will be less easily intelligible and therefore less useful to all except the specialist. Furthermore, the methods actually adopted are justified by the qualitative and often semi-quantitative accuracy of the results.

Quite a number of topics receive a better and more modern treatment than in any other book the reviewer has seen. These include the cell model and average potential model for binary systems; critical phenomena; dipolar effects; and the theory of polymer mixtures. At every stage the comparison of theory and experiment receives due consideration and is illuminated by a large number of graphs and tables. The printing and binding are of the usual high standard of the publishers.

H. S. Green (Dublin).

★ **Seitz, Frederick; and Turnbull, David, editors. Solid state physics. Advances in research and applications. Vol. 3.** Academic Press Inc., New York, 1956. xiv+588 pp. $12.00.

Volume 3 of this series [for Vols. 1 and 2 see MR **19**, 793, 794] contains an assorted fare, with more attention given to structure-sensitive physical properties than in the previous volumes.

In "Group III-group V compounds", H. Welker and H. Weiss give a resumé of the electrical, magnetic and optical properties of the many compounds of this type (e.g. InSb, GaAs, etc.) which have been studied, especially with a view to potential use as semiconductors. In a brief theoretical section, the role of resonance in the binding in these compounds is discussed.

J. D. Eshelby reviews "The continuum theory of lattice defects", an important field in the mathematical theory of elasticity, in which advances are currently being made. This paper gives a general discussion of the principles that are relevant to media with singularities such as point defects or dislocations. Basic theorems about internal stress systems and the energies of interaction of the elastic stress fields of different singularities are derived. In discussing the energy-momentum tensor of the elastic field, the relation of the elastic case to general field theory is considered. In applications to point defects and to dislocations, the value of the use of image fields is emphasized. Recent theories on dislocations in motion and continuous distributions of dislocations are reviewed in sections on more advanced applications.

In "Order-disorder phenomena in metals", L. Guttman deals largely with thermodynamic questions such as the degree ("order") of order-disorder transitions and the calculation of thermodynamic quantities. Comparison is made with experiment wherever possible. There is an introductory section on pair-density functions and their measurement, and a brief final section on kinetics of ordering. This paper is complementary to the paper by T. Muto and Y. Takagi in Vol. 1 of this series.

D. Turnbull begins the chapter "Phase changes" with a resumé of relevant parts of chemical thermodynamics, including the thermodynamics of surfaces. The factors governing the stability of phases are discussed briefly, but the main part of the paper deals with theoretical and experimental work on the kinetics of phase changes, including liquid to solid and solid to solid cases. The main theoretical problems discussed are those of nucleation of the new phase, with which the names of Volmer, Weber, Becker and Döring are usually associated.

In "Relations between the concentrations of imperfections in crystalline solids", F. A. Kroger and H. J. Vink

give a lengthy discussion of the equations for the equilibrium concentrations of imperfections in stoichiometric, non-stoichiometric and impure compounds, and their graphical solution, especially for solid-vapour equilibria. There are many references to experimental determinations.

Finally, C. Kittel and J. K. Galt give a clear and comprehensive review of "Ferromagnetic domain theory", in which many structure-sensitive aspects of ferromagnetism are discussed. The various contributions to the domain energy and the energy of the domain wall are discussed and applied to the deduction of theoretical domain structures. The experimental study of domains is then reviewed. The magnetic properties of small particles are discussed in relation to single domain behaviour and its technical applications in materials of high coercivity. Two final sections deal with the coercive force and the motion of domain walls. *M. S. Paterson.*

Kitaigorodskij, A. I. The theory of the structure amplitude connection and the methods of the direct analysis of the crystal structures. Akad. Nauk SSSR. Kristallografiya **2** (1957), 352–357. (Russian)

The Sayre's law of predominant positivity of the triple product of X-ray structure factors $F_H F_K F_{H+K}$ is discussed. A new method of its application is proposed for direct crystal structure analysis. Contrary to the opinions of previous authors, it is shown that statistical theories are of secondary importance both in proving and applying this law. *V. Vand* (University Park, Pa.).

Vlasenko, V. I.; and Zhdanov, G. S. Automatic synthesis of the two-dimensional pictures of the atomic structures. Akad. Nauk SSSR. Kristallografiya **2** (1957), 358–365 (1 plate). (Russian)

The paper describes a method for conversion of a presentation of a function of two variables into a contour map. Such functions occur in X-ray crystal analysis as two-dimensional Fourier projections of electron density. The function is available in a form of a table of numerical values of the function over a grid of points, obtained from a high-speed digital computer. The method consists first of constructing, by means of electronic equipment, a two-dimensional model of the function. The model is then dissected with a series of planes parallel to the basal plane. The intersections are the contour lines, which can be displayed on a cathode-ray tube screen. Thus an electron density map can be obtained in a few seconds. *V. Vand* (University Park, Pa.).

Cochran, W.; and Douglas, A. S. The use of a high-speed digital computer for the direct determination of crystal structures. I. Proc. Roy. Soc. London. Ser. A. **227** (1955), 486–500.

{Editor's note: this article was listed by title in MR **16**, 527.}

One of the most formidable problems in the analysis of crystal structures by X-ray diffraction is the assignment of relative phases to the structure factors. This paper describes the use of some equality relationships to determine the signs of structure factors in crystals with centro-symmetric projections. Techniques for carrying out the calculations involved using a high speed digital computer are described. The relations used are the following: $S(h)S(h')=S(h+h')$, where $S(h)$ is the sign of the unitary structure factor $U(h)$. An expression giving the probability that this relationship is true is also used.

Such expressions seldom lead to a single set of signs for all the structure factors considered. Usually some ambiguity is found in some of the cases examined, so that the final result consists of a number of sets of signs, only one of which is correct. Any technique which will reduce the number of such sets is valuable, and it is such a technique that is described in this paper. The correct set of signs is that which gives an electron density distribution which is everywhere positive. It is shown that this condition, combined with the assumption of spherically symmetrical atoms, leads to the condition that $\chi=\sum_h \sum_{h'} U(h)U(h')U(h+h')$ should be large and positive. Now χ_e, the expected value of χ, for the correct set of signs can be calculated. Then the authors limit further examination of sets of signs to those for which χ has a value greater than or equal to χ_0, which is taken to be slightly less than χ_e.

This procedure reduces considerably the number of sets of signs which must be examined in detail. However, it still requires the evaluation of χ for all possible sign combinations. The authors then describe a method of reducing the number of χ's which must be evaluated. For, if $Y(h, h')=S(h)S(h')S(h+h')$, then they show that an integer δ may be found such that for all sets $Y(h, h')$ which contain not more than δ negative members, $x \geqq x_0$. The final step consists of drawing Fourier maps for each set of signs. The correct set is then picked out by inspection of the map. Details of programming EDSAC to carry out these calculations are given.

Three compounds for which a correct choice of signs was made by this method are discussed. These are salicylic acid, the intermetallic compound Co_2Al_9, and a complex arsonium bromide. This paper is a dramatic illustration of the potentialities of high speed computers in the solution of the phase problem. *W. M. Macintyre.*

Cochran, W.; and Douglas, A. S. The use of a high-speed digital computer for the direct determination of crystal structures. II. Proc. Roy. Soc. London. Ser. A. **243** (1957), 281–288.

Part I of this paper, reviewed above, described a way of selecting a number of most probable sets of signs which might be applied to structure factors from the large number of possible sets. This technique leads to a small number of most probable sets, and often directly to the correct set. However, this is true only when the number of atoms in the asymmetric unit is quite small or when a few atoms of relatively high atomic number are present. In other circumstances, the number of most probable sets given is quite large, and the selection of the correct set rather more difficult. In the present paper an additional criterion is given which should be applied to the most probable sets of signs.

A function ψ is defined:

$$\psi=\sum_k |\sum_h U(h)U(h+k)|,$$

where $U(h)$ is the unitary structure factor of the plane (h). The sum over h involves only terms for which possible signs are given by the χ criterion of the previous paper. Two values of ψ are calculated. ψ_0 is the value of ψ for which the summation over k involves only those $U(k)$ which are numerically small and below a definite limit in magnitude. ψ_0 should then be quite small for the correct combination of signs. ψ_M is the value of ψ for which the summation over k is confined only to those $U(k)$ which are numerically large; once again the values of $U(k)$ are taken to lie between definite limits. For the correct set of

signs ψ_M should be quite large. The results cited suggest that a low value of ψ_0 is a more reliable indication of the correctness of a set of signs than a high value of ψ_M. Computer techniques for evaluating ψ and also the final Fourier syntheses are discussed. *W. M. Macintyre.*

Bloembergen, N. Spin relaxation processes in a two-proton system. Phys. Rev. (2) **104** (1956), 1542–1547.

The general theory of nuclear spin-lattice relaxation, based on the Boltzmann transport equation for the density matrix, has been formulated in a concise way by Redfield [I.B.M. J. Res. Dev. **1** (1957), 19–31]. This theory is applied to a system of two protons rotating around a given axis. Such proton pairs show resolvable doublets [H. S. Gutowski and G. E. Pake, J. Chem. Phys. **18** (1950), 162–170], whose relaxation processes are investigated in the present paper. The longitudinal and tranverse relaxation times are given. They depend on the angles between the various preferred directions inherent in the problem. Specific expressions for the saturation effects and for the Overhauser effect are also given. Some preliminary experimental support for this theory is listed, and further suggestions in this direction are outlined.

M. J. Moravcsik (Livermore, Calif.).

Bullough, R. Deformation twinning in the diamond structure. Proc. Roy. Soc. London. Ser. A. **241** (1957), 568–577.

Der Verfasser geht von der Auffassung aus, dass bei Zwillingskristallen die Berührungsebene eine aus lauter parallelen und untereinander ähnlichen beweglichen Versetzungen bestehende Ebene ist. Durch Einführung eines Cartesischen Koordinatensystems und einfache rein analytisch-geometrische Betrachtungen wird dann gezeigt, dass die Versetzungsrichtung und die Richtung des Burgersschen Vektors aufeinander senkrecht stehen, die erwähnten Versetzungen also nur Stufenversetzungen sein können. Weiter werden dann Ausdrücke für die Matrix der Rotation des Gitters, für die der Gleitung und ausserdem für die Normale der Zwillingsebene, die alle mit Hilfe eines einzigen Parameters ausgedrückt werden, hergeleitet. Aus dem Werte dieses Parameters bei einem einfachen flächenzentrierten Gitter folgt dann, dass bei dem die Fläche 311 Zwillingsebene sein kann, beim Diamantgitter dagegen, das man sich bekannterweise aus zwei ineinandergeschobenen flächenzentrierten Gittern aufgebaut vorstellen kann, ist die Zwillingsebene mit den niedrigsten Millerschen Indizes die Ebene 123. Damit ist die merkwürdige Erfahrung, dass tatsächlich diese Ebene beim Diamant als Zwillingsebene auftreten kann, theoretisch erklärt. Es sei nur noch bemerkt, dass hier die Berührungsebene auch die Zwillingsebene ist; der Fall dagegen, dass diese zwei Ebenen aufeinander senkrecht stehen würden, ist unter den bezüglich der Versetzungen gemachten Annahmen nicht möglich. *T. Neugebauer.*

Little, W. A. The Overhauser effect in solids. Proc. Phys. Soc. Sect. B. **70** (1957), 785–795.

A theoretical treatment is given of the Overhauser effect in solids containing two or more different nuclear species. In a rigid lattice a decrease is predicted in the macroscopic magnetic moment of one nuclear species on saturating the spin system of the other. In a lattice executing internal motion the Overhauser effect is shown to be strongly dependent upon the value of the lattice correlation time. The usual assumption of the temperature dependence of this correlation time allows one to predict the variation of the Overhauser effect with temperature. *From the author's summary.*

Zheludev, I. S. The symmetry of homogeneous continuous isotropic media in tensor, vector and scalar fields. Akad. Nauk SSSR. Kristallografiya **2** (1957), 334–339. (Russian)

Seventeen point groups of symmetry of homogeneous, continuous isotropic media are obtained for the case when they are in the fields described by polar and axial tensors of the second order. *Author's summary.*

See also: Computing Machines: Černy and Oblonsky. **Quantum Mechanics:** Ishiguro, Kayama, Mizuno, Arai and Sakamoto; Coulson and Kearsley.

Fluid Mechanics, Acoustics

Yih, Chia-Shun. On stratified flows in a gravitational field. Tellus **9** (1957), 220–228.

Among the several loosely connected topics of this paper one finds: an extension of the method of constructing a steady two-dimensional potential flow of a homogeneous liquid, in a gravity field and with a free surface, to the case of two homogeneous liquids separated by an interface; an extension of Rayleigh's stability argument for plane Couette flow, disregarding viscosity effects, to the case of a continuously stratified liquid.

G. Kuerti (Cleveland, Ohio).

Glauert, M. B. The flow past a rapidly rotating circular cylinder. Proc. Roy. Soc. London. Ser. A. **242** (1957), 108–115.

The flow considered is that about a rotating circular cylinder in an incompressible, viscous, steady stream, in the regime where the circulation is so great as to produce a stagnation point in the fluid and closed streamlines about the cylinder. Prandtl [Naturwissenschaften **13** (1925), 93–108] conjectured that the circulation would never reach such great values and gave an argument based on shedding of vorticity from the boundary layer. Experiments have not exhibited such values, but the author believes they are inconclusive. He also offers some arguments in opposition to Prandtl's conjecture.

The author sets up the boundary-layer equations for this case and constructs a solution in series form; it is, of course, periodic in the angular coordinate around the cylinder. The result relates the circulation and thus the lift coefficient to the peripheral speed. There is also a torque, and the author discusses the interesting angular-momentum balance. *W. R. Sears* (Ithaca, N.Y.).

van Spiegel, E. Theory of the circular wing in steady incompressible flow. Nat. Luchtvaartlab. Amsterdam. Rep. NLL-TN F. **189** (1957), 52 pp. (5 plates).

Timman's method [J. Aero. Sci. **21** (1954), 230–236, 250; MR **15**, 662] is applied to the problem stated in the title. The principal motivation seems to be the doubts cast on earlier investigations of the circular wing by more recent workers. The present method is to use oblate spherical coordinates, formed by rotating confocal ellipses about their minor axes, in which harmonic solutions appear as sums of Legendre functions. Regular solutions for both the velocity potential and the acceler-

ation potential are written; one represents circulation-free flow and the other flow with "shock-free entry," for the same wing geometry. The former exhibits the square-root singularity all around the edge of the planform. By comparison of the two, a singular solution is found which, added to the regular acceleration potential, satisfies the Kutta-Joukowski condition at the trailing edge and introduces a singularity at the leading edge.

The resulting solution, for given geometry, poses an integral-equation problem, which is numerically solved by truncating the series. As examples, a flat plate at incidence and a wing with parabolic camber (downwash varying linearly with streamwise coordinate) are treated. Lift and moment are computed and compared with results of five earlier investigations. As a check on the accuracy of the present work, a reverse-flow theorem due to Flax [ibid. **19** (1952), 361–374; MR **14**, 218] is applied; it is stated that the theorem is accurately satisfied.

There are several appendices. In one of these, a disagreement with the theory of Küssner [ibid. **21** (1954), 17–26, 36; **22** (1955), 227–230; Z. Flugwiss. **4** (1956), 21–26; MR **15**, 480; **16**, 878; and Proc. 2nd European Aero. Congress (Scheveningen, 1956) (to be published)] is mentioned. In the numerical results mentioned above, the agreement with Küssner is poor. *W. R. Sears.*

Rethorst, Scott. Aerodynamics of nonuniform flows as related to an airfoil extending through a circular jet. J. Aero. Sci. **25** (1958), 11–28.

The problem considered is the evaluation of the lift coefficient of a wing which extends through a circular jet. The method employed is the investigation of the effect of the jet on elementary horseshoe vortices placed inside and outside the jet. The effect of the jet is split into two parts, one of which is even and the other odd, with respect to the axial co-ordinate. Both parts are included in the present theory, in contrast to that of Koning, which omitted the contribution of the odd part. A simple spanwise integration is then required to calculate results for a general wing. The present theory is compared with Koning's and with experiment; it is in good agreement with experimental results whereas Koning's theory is not. *G. N. Lance* (Southampton).

Germain, Paul. Sur le minimum de traînée d'une aile de forme en plan donnée. C. R. Acad. Sci. Paris **244** (1957), 1135–1138.

A previous result of Nikolsky [Ninth Congr. Appl. Mech. (1956)] and Ward [Aero. Res. Comm. Rep. 18, 711 (1956)] for the minimum drag of a lifting wing of given planform is generalized with the aid of Hilbert space theory and applied to a delta wing having prescribed lift and moment coefficients. *J. W. Miles.*

Germain, Paul. Aile symétrique à portance nulle et de volume donné réalisant de minimum de traînée en écoulement supersonique. C. R. Acad. Sci. Paris **244** (1957), 2691–2693.

Results analogous to those in the above paper are obtained for a non-lifting wing of given volume. Explicit results are limited to planforms having an axis of symmetry. *J. W. Miles* (Los Angeles, Calif.).

Soundranayagam, S. The secondary flow behind a cascade. J. Aero. Sci. **24** (1957), 706–707.

J. H. Preston [Aero. Quart. **5** (1954), 218–234] applied the vortex filament concept to obtain expressions for the secondary flow downstream of an impulse cascade. In the present paper this method is used for diffusing and accelerating cascades. For the latter case, results are obtained which are in agreement with those differently derived by W. R. Hawthorne [Quart. J. Mech. Appl. Math. **8** (1955), 266–279; MR **17**, 423]. *M. Marden.*

Jacobs, Willi. Neuere theoretische Untersuchungen über den Strahlflügel in zweidimensionaler Strömung. Z. Flugwiss. **5** (1957), 253–259.

A review article.

Pyhteev, G. N. Kirchhoff's discontinuous bounded flow past a family of curves. Dokl. Akad. Nauk SSSR (N.S.) **113** (1957), 513–516. (Russian)

Vorovič, I. I.; and Yudovič, V. I. Impact of a circular disk on a liquid of finite depth. Prikl. Mat. Meh. **21** (1957), 525–532. (Russian)

The problem of the normal or oblique impact of a circular disk on the surface of an incompressible fluid layer is formulated as a boundary value problem for the Laplace equation in cylindrical coordinates. Explicit solutions are found with the aid of Fourier transforms and the solution of an integral equation by iteration. Conditions are given for oblique impact without breakage of the surface. *W. Kaplan* (Ann Arbor, Mich.).

Crapper, G. D. An exact solution for progressive capillary waves of arbitrary amplitude. J. Fluid Mech. **2** (1957), 532–540.

An exact solution is found in a fairly simple form for two-dimensional progressive waves of arbitrary amplitude on a fluid of unlimited depth, when only surface tension and not gravity is taken into account as the restoring force. The calculated wave forms are exhibited graphically for various amplitudes, and the relation between wave velocity and amplitude is plotted. The wave of greatest height occurs when the vertical distance between trough and crest is 0.730 wavelengths (compared with 0142 for gravity waves). Higher waves are prevented from appearing by the enclosing of air bubbles in the troughs. (Author's summary.) *A. E. Heins* (Pittsburgh, Pa.).

Portnov, I. G. Concerning the leading edge of a region of cavitation. Izv. Akad. Nauk SSSR. Otd. Tehn. Nauk **1957**, no. 3, 99–107. (Russian)

The maintenance of cavitation is formulated as a problem in stability of a surface separating a liquid from a mixture of liquid and gas.. The problem is considered locally, the fluid motion being considered as one-dimensional. Equations are postulated for heat conduction and convection, with continuity conditions at the transition surface. An equilibrium solution is found and its stability studied by standard methods. In particular, sufficient conditions are given for instability — i.e., collapse of cavitation — in terms of the thermodynamic parameters. *W. Kaplan* (Ann Arbor, Mich.).

Slezkin, N. A. On the development of the flow of a viscous fluid between parallel porous walls. Prikl. Mat. Meh. **21** (1957), 591–593. (Russian)

The author uses the Laplace transformation to obtain a solution of linearized Navier-Stokes equations, corresponding to a flow between parallel flat plates in which fluid is being absorbed by the plates at a rate proportional to the difference between the local pressure and a fixed pressure. *R. Finn* (Pasadena, Calif.).

Mallick, D. D. Nonuniform rotation of an infinite circular cylinder in an infinite viscous liquid. Z. Angew. Math. Mech. **37** (1957), 385–392. (German, French and Russian summaries)

The unsteady motion is considered in the cases when the liquid is either inside or outside a circular cylinder that is suddenly caused to rotate with constant angular velocity. Analytical solutions are obtained by Laplace transform techniques, and numerical results are presented in the form of graphs and tables illustrating the approach to the steady state. *D. W. Dunn* (Ottawa, Ont.).

Faĭnzil'ber, A. M. Similarity integrals of vortex and temperature fields. Dokl. Akad. Nauk SSSR (N.S.) **100**, 225–228 (1955). (Russian)

If heat is released on the surface of a profile, the equations of the hydrodynamics of viscous media have a similarity integral of velocity and temperature fields. But this integral is valid only for flow past a plate and only for constant temperature of the surface. In the present article, other integrals are obtained, valid for flow past a profile of arbitrary shape and for variable temperature of the surface. *From the introduction.*

Chester, W. An exact solution of the Navier-Stokes equations. J. Aero. Sci. **24** (1957), 853–854.

There is an exact solution of the equations of motion and heat conduction for a perfect gas, if the velocity is $\{u(y), 0, 0\}$, the pressure is $p(x, t)$, the density is constant, and $\gamma=1.5$; if $\gamma=2$, there is a similar solution appropriate to flow through a circular pipe.

W. R. Dean (London).

Cohen, Nathaniel B. A power-series solution for the unsteady laminar boundary-layer flow in an expansion wave of finite width moving through a gas initially at rest. NACA Tech. Note no. **3943** (1957), 56 pp.

The laminar boundary layer obtained when an expansion wave advances along a plane wall into a viscous and heat conducting fluid is considered in terms of the boundary layer equations of motion and energy. Solutions are obtained in the form of power series in a similarity coordinate. The first three terms are numerically evaluated and velocity and enthalpy profiles in the boundary layer and the local skin friction and heat transfer at the wall are obtained. These numerical results are valid near the leading edge of the wave. A comparison is made with the results of an analysis of Mirels [NACA Tech. Note no. **3712** (1956)] in which the finite width expansion wave is considered as a line discontinuity. The two solutions are shown to be significantly different. *P. Chiarulli.*

Trimpi, Robert L.; and Cohen, Nathaniel B. An integral solution to the flat-plate laminar boundary-layer flow existing inside and after expansion waves and after shock waves moving into quiescent fluid with particular application to the complete shock-tube flow. NACA Tech. Note no. **3944** (1957), ii+180 pp.

The unsteady two-dimensional laminar boundary-layer flow inside and behind a centered expansion wave or behind a shock wave is studied in terms of a Kármán-Pohlhausen integral procedure. Based on an assumption of a conical flow in the outer-inviscid flow (valid for a shock tube flow), a similarity variable, x/t, is introduced into the integrated form of the boundary-layer equations. The resulting differential equations are integrated by a method of characteristics for the shock wave case and by a numerical procedure (five-term and six-term Pohlhausen expansions for the velocity and enthalpy profiles) in the rarefaction wave cases.

Specific calculations for the complete boundary layer flow are carried out for air-air, helium-air, and hydrogen-air shock tubes. Results are in general agreement with those of H. Mirels [NACA Tech Note no. **3401** (1955); **3278** (1956); **3712** (1956); MR **16**, 759; **18**, 355].

P. Chiarulli (Chicago, Ill.).

Săvulescu, St. Une méthode simple pour l'étude des caractéristiques de la couche limite. Rev. Méc. Appl. **1** (1956), no. 2, 37–42.

The equations for two-dimensional boundary-layer flow of a compressible fluid, subjected to the von Mises transformation, may be written as

$$\frac{\partial}{\partial \eta_\delta}\tau=L\left(\frac{\partial u_a}{\partial \xi},\frac{\partial u_a}{\partial \eta_\delta}\right),\quad \frac{\partial}{\partial \eta_\Delta}(\tau u+q_y)=L_1\left(\frac{\partial E_a}{\partial \xi},\frac{\partial E_a}{\partial \eta_\Delta}\right).$$

Here τ=shear stress, q_y=heat flow in y-direction; u_a and η_δ, E_a and η_Δ are, respectively, x-velocity and streamfunction, total energy and streamfunction, all dimensionless so that they may be zero at the wall and unity at the kinematic and thermal boundary layer edges (δ, Δ); ξ is the dimensionless x-coordinate; and L, L_1 denote linear functions.

If $u_a=\eta_\delta^n$ and $E_a=\eta_\Delta^m$ (where m and n may depend on x) are introduced as first approximations, τ and $\tau u+q_y$ are obtained by a quadrature. Another quadrature then gives improved values of u_a and E_a, where $\tau \propto \partial u_a/\partial \eta_\delta$ and $\tau u+q_y \propto \partial E_a/\partial \eta_\Delta$. (The last relation seems, however, to presuppose that the Prandtl number be unity.) Three examples are discussed briefly. *G. Kuerti.*

Gupta, A. S. Advancement of a compressible heat conducting fluid over an infinite flat plate. Z. Angew. Math. Mech. **37** (1957), 349–353. (German, French and Russian summaries)

The equations of motion in the unsteady case when a compressible viscous fluid with vanishing pressure gradient flows past an infinite flat plate have been integrated in semi-convergent expressions when the physical constants depend on temperature. Simple expressions are obtained for temperature and velocity distribution in the boundary layer, and for the drag coefficients and their dependence on physical constants. (From the author's summary.) *R. C. DiPrima* (Troy, N.Y.).

Voronin, V. I. On asymptotic solution of the laminar-boundary-layer equations for a flat plate. Trudy Voronež. Gos. Univ. Fiz.-Mat. Sb. **33** (1954), 63–69. (Russian)

Gil, G. V.; and Myshkis, A. D. Asymptotic behaviour of solutions of a non-linear boundary problem in the boundary layer theory. Dokl. Akad. Nauk SSSR (N.S.) **112** (1957), 599–602. (Russian)

Let $y(t)$ satisfy the differential equation $y'''+2yy''+2\beta(k^2-y'^2)=0$ for $0\leq t<\infty$; here β and k are constants, $\beta\geq 0$, $k>0$. Let $y(0)=y'(0)=0$, $y'(\infty)=k$. [See S. Goldstein, Modern developments in fluid dynamics, vol. I, Oxford, 1938, p. 139.] The authors obtain the asymptotic formula for $t\to\infty$: $y=kt-C+t^{-2-2\beta+o(1)}\exp(-kt^2+2Ct)$ where C is a positive constant; similar formulas are obtained for y', y''. The proofs are elementary but ingenious.

W. Kaplan (Ann Arbor, Mich.).

Szaniawski, Andrzej. Propagation of small perturbations in a gas-liquid emulsion. Rozprawy Inż. **5** (1957), 269–329. (Polish. Russian and English summaries)

Under simplifying assumptions such as a quasi-homogeneous state of gas-liquid mixture, uniform distribution, etc., the author derives and linearizes equations of motion. Their number is less by two than the number of unknown functions. Three types of additional assumptions are considered: barotropic changes, disregarding the influence of inertia of liquid surrounding a gas bubble and harmonic vibration of a gas bubble in an infinite incompressible fluid. In each case the author calculates wave velocity and damping coefficient. A resonance phenomenon can be also observed. *M. Z. v. Krzywoblocki* (Urbana, Ill.).

Ebert, Rolf. Zur Instabilität kugelsymmetrischer Gasverteilungen. Z. Astrophys. **42** (1957), 263–272.

★ **Pai, Shih-I. Viscous flow theory. II. Turbulent flow.** D. Van Nostrand Company, Inc., Princeton, N.J.-Toronto-New York-London, 1957. xi+277 pp. $6.75.

This is the second of a two-volume treatment of the dynamics of viscous fluids [for v. 1, 1956, see MR **18**, 437], and covers the various aspects of turbulent flow. After an introductory chapter (chapter I) on the fundamentals of turbulent flow, the author devotes about half of the volume to the semi-empirical theories of turbulence, (chapter II) with the various applications of such concepts to turbulent flow in pipes and channels (chapter III), turbulent flow over a flat plate (chapter IV), turbulent boundary layer with a pressure gradient (chapter V), turbulent boundary layer of a compressible fluid (chapter VI), and turbulent jet mixing regions and wakes (chapter VII). Engineers and other people interested in applications will find this part of the book most useful.

The fundamentals of the statistical theory of turbulence are discussed in chapters VIII and IX. This is followed by a chapter on turbulent diffusion. The following two chapters deal with the dynamics of homogeneous turbulence. This is followed by a chapter on locally isotropic turbulence and non-isotropic turbulence. The final chapter deals with turbulence in compressible flow and magneto-hydrodynamics.

The book gives a very good account of the overall picture of the theory of turbulence, fundamental as well as semi-empirical. However, in such an extensive treatise of a subject as complex as the present one, it is not difficult to find differences of opinion of what should be included. It appears to the present reviewer that the omission of all reference to the recent work of Proudman and Reid [Philos. Trans. Roy. Soc. London. Ser. A. **247** (1954), 163–189; MR **16**, 299], and of Tatsumi [Proc. Roy. Soc. London. Ser. A. **239** (1957), 16–45; MR **18**, 694] gives an unsatisfactory picture of the theory of decaying turbulence. A similar remark applies to the omission of any reference to Chandrasekhar's recent papers [e.g. Phys. Rev. (2) **102** (1956), 941–952]. It is possible that the manuscript of this book was largely completed before these papers appeared (this is certainly true in the case of Tatsumi's paper), but perhaps it is not out of place to bring up this point here. Another point showing the difficulty of writing an account of the ever-changing subject of the theory of turbulence is the change of opinion on the "Loitsiansky invariant" [see Batchelor and Proudman: Philos. Trans. Roy. Soc. London. Ser. A. **248** (1956), 369–405; MR **18**, 843]. It is also worth consideration whether some reference should have been made to the study of the acoustic effects of turbulent motion. However, despite these omissions, readers interested in the subject of turbulence will find this book a very handy reference. The fairly complete bibliography at the end of each chapter also enhances the value of this book.

C. C. Lin (Cambridge, Mass.).

Backus, George. The existence and uniqueness of the velocity correlation derivative in Chandrasekhar's theory of turbulence. J. Math. Mech. **6** (1957), 215–233.

The basic equation underlying the theory of turbulence considered in this paper is

$$(1)\qquad x(9+4x^2\phi)\phi'''+(9-6x^2\phi)\phi''-44x\phi\phi'+44\phi^2=0,$$

where primes denote differentiation with respect to x; and the function which is relevant in the physics of the theory is

$$(2)\qquad \sigma(x)=\tfrac{3}{8}(\phi-x\phi').$$

The question which is considered in this paper is this: How many solutions has (1) which are analytic except at the fixed and the possible movable singular points of (1) and which provide via (2) continuous differentiable functions which satisfy the conditions

$$(3)\qquad \sigma(x)=\sigma(-x),\ \lim_{x\to\infty}\sigma(x)=0 \text{ and } \sigma(0)>0?$$

To answer this question the author investigates the nature of the solution of equation (1) near each of the fixed singular points $x=0$, $9+4x^2\phi=0$ and $x=\infty$. The theorems which are established pertinent to these three singularities are the following: (A) The singularity at $x=0$: σ is one of a family of functions determined via (2) from the homologues $a^2\phi(ax)$ (where a is a constant) of a unique analytic function $\phi(x)$ satisfying (1), $\phi(0)=9/2$ and $\phi'(0)=0$ and which is given by the power series

$$\tfrac{3}{8}\phi(x)=\sum_{n=0}^{\infty}a_nx^n,\ a_0=1,$$

$$(4)\qquad a_{n+2}=-\sum_{r=0}^{n}a_ra_{n-r}\frac{(r-1)(r+2)(2r-11)}{(n+2)(n+1)^2},$$

which is uniformly convergent in a finite neighborhood of $x=0$. (B) ϕ has a one parameter family of continuations across the singular hyperbola $9+4x^2\phi=0$, all of which make ϕ, ϕ' and ϕ'' continuous at the crossing point. Also, at every point (x_0, ϕ_0) on the singular hyperbola there is a one parameter family of analytic solutions of (1) one for each slope ϕ_0'. (C) The singular point at $x=\infty$: To study the behavior of the solution at $x=\infty$, the transformation $g(z)=z\phi(z^{-1})$ is made and the resulting equation

$$(5)\qquad z^3(4g+9z^3)g'''=-18z(z^3+g)g''+44gg'$$

is studied. If $g(z)$ is a solution of (5) then so is $a^3g(z/a)$. So without loss of generality we may assume $g(0)=0$ or $g(0)=1$. If $g(0)=1$, then the author shows that every solution of equation (5) which is continuously differentiable at $z=0$ and $g(0)=1$ and $g'(0)=0$ has the form

$$g(z)=1+\tfrac{1}{3}Cz^3+\tfrac{1}{9}z^3m(z, C),$$

where $m(z, C)$ is continuously differentiable in a closed neighborhood of $z=0$, vanishes at $z=0$ and is uniquely determined by C. On the basis of the foregoing results the behavior of the solutions of (1) in the large is examined.

The author summarizes the results of his paper as follows: "It can be proved from the physics of homogeneous isotropic turbulence that $\sigma(x)=\sigma(-x)$. [See S. Chandrasekhar, Phys. Rev. (2) **102** (1956), 941–952.] The

assumptions $\sigma(0)>0$ and $\lim_{x\to\infty}\sigma(x)=0$ are physically reasonable because $\sigma(0)$ is essentially the means square of the vorticity and $\lim_{t\to\infty} r^4\chi(r,t)=0$ says essentially that the velocities at two different points r_1 and r_2 at times t_1 and t_2 become statistically independent as $|t_2-t_1|\to\infty$. If one imposes the further restriction that $\sigma'(x)$ be continuous, Chandrasekhar's theory of turbulence leads, in the case of zero viscosity, to a specification of the radial derivative of the second order velocity correlation which is unique up to a homology transformation. If one cannot demand that $\sigma'(x)$ be continuous at $x=0$, the uniqueness theorem proved in this paper fails, and many more solutions of equation (1) become acceptable".

S. Chandrasekhar (Williams Bay, Wis.).

Keune, Friedrich. Reihenentwicklung des Geschwindigkeitspotentials der linearen Unter- und Überschallströmung für Körper nicht mehr kleiner Streckung. Z. Flugwiss. **5** (1957), 243–247.

First, a simple derivation of the expression for the velocity potential of subsonic flow about a non-lifting body in familiar slender-body theory is given and an expression for the error compared to the complete linear small-perturbation theory is obtained. By treating the remainder terms in similar fashion, i.e., assuming the lateral extent of the body to be small compared to the streamwise dimension, the author obtains the second-order terms, and so forth. This process is carried out in detail through the third-order terms. The analogous process for supersonic flow is carried through the second-order terms only, including an expression for their error, but the next terms can apparently be written down by analogy with the subsonic result. All the results are in accord with the author's earlier, general discussion [Z. Flugwiss. **5** (1957), 121–125; MR **19**, 89 . The flow always consists of a two-dimensional "cross-section effect" plus a "special influence". *W. R. Sears.*

Ergun, A. N. Two dimensional wave motion in a compressible rotating fluid bounded internally by a radially oscillating circular cylinder. Comm. Fac. Sci. Univ. Ankara. Sér. A. **8** (1956), 6–26. (Turkish summary)

In a recent paper Davies [MR **11**, 474] has presented an alternative theory to the usual linearised theory of unsteady compressible flow by superimposing a small perturbation upon the correct basic irrotational flow pattern. The present paper is concerned with a particular example of this theory in which a rotating compressible fluid is bounded internally by a radially oscillating circular cylinder. *Author's summary.*

Korobeinikov, V. P.; and Melnikova, N. S. On exact solutions of linearized problem on point explosion with counterpressure. Dokl. Akad. Nauk SSSR (N.S.) **116** (1957), 189–192. (Russian)

Andriankin, È. I.; and Ryžov, O. S. Propagation of a thermal nearly spherical wave. Dokl. Akad. Nauk SSSR (N.S.) **115** (1957), 882–885. (Russian)

Schubert, Hans; und Schincke, Erich. Zum Konturproblem der Hodographenmethode im Unterschall. Ber. Verh. Sächs. Akad. Wiss. Leipzig. Math. Nat. Kl. **102** (1957), no. 2, 25 pp.

The problem considered is that of subsonic flow without circulation past a body of prescribed shape and symmetric about two perpendicular axes. A simple linear equation is obtained by transforming to hodograph variables and using a quadratic approximation to the perfect gas relations. The boundary conditions are also transformed but they have to be applied on a curve which is unknown at this stage of the solution process. The authors propose a method of overcoming this difficulty but the details are left for a later paper. *J. J. Mahony.*

Helliwell, J. B.; and Mackie, A. G. Two-dimensional subsonic and sonic flow past thin bodies. J. Fluid Mech. **3** (1957), 93–109.

This is a well-written paper, which includes a valuable introductory review of the problem of near-sonic flow past thin bodies. It is of interest also to the analyst for its manipulations of Bessel-function series and integrals. The reviewer cannot improve on the authors' summary:

"Hodograph methods are applied to determine the flow at high subsonic and sonic velocities past two-dimensional, thin, symmetrical bodies. The boundary value problem for the determination of the stream function, which in the present theory is a solution of Tricomi's equation, is simplified by the assumption of a free stream breakaway at sonic velocity from the shoulder of the body. A solution is obtained in terms of Bessel functions.

"The flow past a wedge of small angle is discussed and expressions are obtained for the pressure on the nose, the drag coefficients and the width of the wake. A comparison with the corresponding results in the case of sonic velocity derived by Guderley Yoshihara [J. Aero. Sci. **17** (1950), 723–735; MR **15**, 264] shows that the present simpler theory yields very similar values for the pressure over the nose.

"Finally the flow at sonic velocity past a profile which is a first-order perturbation upon a wedge profile is analysed on the basis of the same free streamline theory. The flow pattern is obtained past an arbitrary specified body by an application of the Hankel inversion theorem and an expression is deduced for the drag".

T. M. Cherry (Melbourne).

Aslanov, S. K. Resistance of a wedge-shaped profile in a stream of sonic velocity. Prikl. Mat. Meh. **20** (1956), 756–760. (Russian)

Ryhming, I. Axiale Rückwirkungen von Überschallschaufelgittern. Z. Angew. Math. Mech. **37** (1957), 370–385. (English, French and Russian summaries)

Da die Rückwirkung eines Überschallgitters auf die stromaufwärts liegende Strömung nur möglich ist, wenn die Machzahl der Geschwindigkeitskomponente senkrecht zum Gitter kleiner als 1 ist, wird nur diese Strömungsform behandelt. Für das Gitter ohne vorgeschaltetes Leitrad wird gezeigt, dass Machzahl und Richtung der Zuströmung nicht beliebig gewählt werden können, sondern dass nur eine von der Gitteranordnung und der Umfangsgeschwindigkeit abhängige Kombination beider Grössen möglich ist. Wird ein Leitrad vorgeschaltet, dann ergibt eine quasistationäre Betrachtung der Strömung, dass bei einigen Anordnungen von Leit- und Laufrad die lineare Theorie keine brauchbaren Lösungen mehr liefert. Eine Lösung ist lediglich mit dem exakten Charakteristikenverfahren möglich. Eine Erweiterung der quasistationären Betrachtung auf die tatsächlichen Verhältnisse bewegter Überschallgitter wird angekündigt.

L. Speidel (Mülheim).

Van Dyke, M. D. A model of supersonic flow past blunt axisymmetric bodies, with application to Chester's solution. J. Fluid Mech. 3 (1958), 515–522.

Cet article constitue une nouvelle tentative pour résoudre le problème de l'onde de choc détachée, dans le cas d'un écoulement permanent à symétrie axiale. La fonction de courant est exprimée en coordonnées paraboliques ξ, η, l'onde de choc étant la surface coordonnée $\eta=1$; un développement limité au second ordre est effectué suivant les puissances croissantes de $1-\eta$. Cette méthode est comparée à celle de Chester [même J. 1 (1956), 353–365, 490–496; MR 19, 353] qui consiste à développer la fonction de courant suivant les puissances entières négatives du nombre de Mach amont et suivant les puissances entières positives du paramètre $\delta=(\gamma-1)/(\gamma+1)$, γ étant l'indice adiabatique. Dans la méthode de Chester, la convergence est lente; les raisons apparaissent dans la comparaison des deux méthodes. *H. Cabannes.*

Rościszewski, Jan. Methods for the analysis of the interaction between a shock wave and a simple rarefaction wave. Rozprawy Inż. 5 (1957), 241–268. (Polish. Russian and English summaries)

After giving an analysis of the fundamentals of the wave interaction problem in a one-dimensional non-steady flow, the author discusses the methods of solving the boundary value problem by using difference equations and the following methods: (i) transformation into velocity-sound velocity plane; (ii) Friedrichs' method with some modifications; (iii) Geiringer's Lagrangian representation. J. v. Neumann's model is also mentioned.

M. Z. v. Krzywoblocki (Urbana, Ill.).

Rogers, M. H. The isothermal expansion of a gas cloud into a non-uniform atmosphere. Arch. Rational Mech. Anal. 1 (1957), 22–34.

A similarity solution is found for the problem of a sphere expanding at a uniform rate in a non-uniform gas, assuming that all changes are isothermal (even across shocks), the initial density distribution satisfies a power law, and the field of force (gravity) maintaining the density distribution is neglected. The solution is of the Taylor type with flow variables functions of radial distance divided by the time, and the flow is headed by a shock moving with constant velocity. The application is to the formation of O-stars in interstellar gas clouds. Reference is made to Savedoff and Greene [Astrophys. J. 122 (1955), 477–487], who treated this problem for uniform density, and the results are compared. (Savedoff and Greene justify the assumption of isothermal changes for this application.) *G. B. Whitham* (New York, N.Y.).

Meyer, R. F. The impact of a shock wave on a movable wall. J. Fluid Mech. 3 (1957), 309–323.

A solution is found under the assumption that the flow is one-dimensional and isentropic. A check is provided by a physical argument which shows that the transmitted shock ultimately attains the strength of the incident shock, while the reflected shock decays to a sound wave. The agreement with experimental results is good.

H. C. Levey (Melbourne).

Korobeinikov, V. P. On propagation of strong spherical blast wave in gas with heat conduction. Dokl. Akad. Nauk SSSR (N.S.) 113 (1957), 1006–1009. (Russian)

Westervelt, Peter J. Scattering of sound by sound. J. Acoust. Soc. Amer. 29 (1957), 199–203.

In this important paper, the author has studied the modifications to a given three-dimensional sound field, due to the retention of terms in the equations of motion of the order of the squares of the disturbances, together with the neglect of viscosity, heat conduction and terms of the order of the cubes of the disturbances. Using an approach of the reviewer [Proc. Roy. Soc. London. Ser. A. 222 (1954), 1–32; MR 15, 837], he writes the equation of motion with the first-order "wave equation" terms on the left and all the second-order terms on the right. By using the first-order acoustic relations many times to modify the form of the right-hand side (thus introducing only errors of an order of magnitude already neglected), he obtains the equation

$$\Box^2\rho_s=c_0^{-2}\{\Box^2E-\nabla^2(2T+kV)\}. \tag{1}$$

Here, ρ_s is the second-order density field (which must be added to the first-order density field, which satisfies the ordinary acoustic equations, to give the complete density field to the approximation here used). T is the kinetic-energy density in the sound wave, V the potential-energy density, $E=T+V$, c_0 is the sound speed in the undisturbed fluid, and k is a constant which for a perfect gas with constant adiabatic index γ is $(\gamma-1)$. Finally, ∇^2 is the Laplacian and $\Box^2=\nabla^2-c_0^{-2}\partial^2/\partial t^2$ the d'Alembertian.

An interesting form in which (1) can be cast, not mentioned in the paper, is

$$\Box^2(p_s-p_R)=-c_0^{-2}\frac{\partial^2}{\partial t^2}(2T+kV), \tag{2}$$

where $p_R=V-T$ is the radiation pressure. This shows that the second-order pressure field consists of the radiation pressure together with the field of a distribution of simple sources. It would be interesting to investigate the latter field, in the case of a sound beam confined in extent, in the region outside the beam where p_R is negligible.

However, the author's main concern is with the interaction of two sound beams, each simple-harmonic with (radian) frequencies ω_1, ω_2. In this case, he writes energies like T as the sum of terms T_{11} quadratic in the amplitude of the first sound beam, T_{22} quadratic in that of the second, and T_{12} bilinear in the two amplitudes. Replacement of T, V by T_{12}, V_{12} on the right of (1) yields the equation for the "scattered sound" (of frequencies $\omega_1+\omega_2$ and $|\omega_1-\omega_2|$) due simply to the interaction of the two beams. The new equation is then recast as

$$\Box^2\Big\{\rho_s-\frac{V_{12}-T_{12}}{c_0^2}+\frac{k(\omega_1^2+\omega_2^2)}{4c_0^2\omega_1^2\omega_2^2}\Big[\frac{\partial^2V_{12}}{\partial t^2}+(\omega_1^2+\omega_2^2)V_{12}\Big]\Big\}=-\frac{2+k}{c_0^4}\frac{\partial^2T_{12}}{\partial t^2} \tag{3}$$

(the exponent $\frac{1}{2}$ on the square bracket in the paper being a misprint). Equation (3) shows that, for two beams perfectly at right angles (so that $T_{12}=0$ everywhere), the expression in braces is zero. In this case ρ_s vanishes outside the interaction region. On the other hand, Ingard and Pridmore-Brown [J. Acoust. Soc. Amer. 28 (1956), 367–369] attempted both a theoretical and an experimental treatment of this case, and obtained non-negligible scattered radiation by both treatments, although the agreement between the two was poor.

The reviewer has had the advantage of Pridmore-Brown's collaboration in checking the theory of this paper; both of us agree that it is correct, and that the author correctly explains the wrong theoretical results of

Ingard and Pridmore-Brown by referring to the singularities in the assumed primary fields. The author's explanation of their experimental results as due to radiation pressure, is, however, untenable, since the microphone was much farther outside both beams than he supposes. Accordingly, imperfect perpendicularity of the two beams near their edges seems the most probable explanation. *M. J. Lighthill* (Manchester).

Rocard, Yves. Propagation du son dans un vent variable. C. R. Acad. Sci. Paris **244** (1957), 1339–1341.

Author studies the propagation of sound in a stream which is flowing non-uniformly in the direction of sound propagation. He finds that the non-uniform velocity does not affect the sound frequency but that the wave amplitude increases or decreases exponentially according as the velocity gradient is negative or positive. The reviewer cannot accept the results in the present form for many reasons, among which are: unclear assumptions and linearizations, assumption of a constant speed of sound and the non-consistent use of the non-uniformity of the stream velocity. *P. Chiarulli* (Chicago, Ill.).

Lamb, George L., Jr. On the transmission of a spherical sound wave through a stretched membrane. J. Acoust. Soc. Amer. **29** (1957), 1091–1095.

The transmission of a spherical sound wave through a homogeneous stretched membrane of infinite extent is investigated theoretically. An integral representation of the transmitted sound field is initially derived. The path of integration is then transformed into the complex plane and the integration carried out in an approximate manner by the method of stationary phase.

The transmitted sound field is found to be composed of two parts, an outgoing spherical wave modified by an amplitude factor containing angular dependence and a surface wave. The surface wave, which results from the free flexural vibration of the membrane itself, exhibits an interesting "zone of silence" in the transmitted sound field. *Author's summary.*

Aver'yanov, S. F.; and Cyul, Sin-E. On the calculation of drainage in the presence of infiltration. Izv. Akad. Nauk SSSR. Otd. Tehn. Nauk **1957**, no. 3, 115–124. (Russian. Chinese summary)

The authors derive various expressions for the discharge and other quantities related to the type of problems mentioned in the title. These expressions contain elliptic functions. However, because these functions are not properly tabulated, the authors do not present the warranted numerical results. It seems to the present reviewer that the authors could have presented known approximations which exist in the literature. Also, these elliptic functions are not difficult to compute, particularly for the present set of values for the parameters involved. An example is presented to illustrate the authors' method. *K. Bhagwandin* (Oslo).

Cickišvili, A. R. Filtration from a channel with trapezoidal cross-section. Izv. Akad. Nauk SSSR. Otd. Tehn. Nauk **1957**, no. 3, 125–133. (Russian)

The author employs a method outlined in a monograph of B. K. Rizenkampf [Uč. Zap. Saratov. Gos. Univ. **15** (1940), no. 5, 3–93 (not available to the present reviewer); MR **11**, 62] to study filtration in channels with trapezoidal cross-sections. Complex-function analysis is employed. Expressions for the free-surface and the discharge are presented in terms of infinite series. However, as the author notes, it seems difficult to establish rigorously the regions of validity of the obtained expressions.

The author does not seem to be aware of the fact that the present problem has also been dealt with by A. M. Mhitaryan [Ukrain. Mat. Ž. **6** (1954), 448–456; MR **17**, 1147] as well as by the Bucharest group (a number of these papers are reviewed in Mathematical Reviews).

K. Bhagwandin (Oslo).

Cickišvili, A. R. Semi-inverse method in the theory of filtration from curvilinear channels. Izv. Akad. Nauk SSSR. Otd. Tehn. Nauk **1957**, no. 4, 129–133. (Russian)

The author obtains closed-form expressions for the free-surface of the fluid in curvilinear channels subject to filtration processes. The method employed is that of M. A. Lavrent'ev [cf Lavrent'ev and Sabat, Methods of the theory of functions of a compex variable, Gostehizdat, Moscow-Leningrad, 1951; MR **14**, 457] in the theory of conformal mapping. The author's treatment of curvilinear boundaries is straightforward. However, in the absence of any appraisal of the error involved in the approximations — this is usually rather important in the application of Lavrent'ev's methods — the analysis cannot be considered rigorous. *K. Bhagwandin* (Oslo).

Cickišvili, A. R. On the iteration method of N. M. Gersevanov. Prikl. Mat. Meh. **21** (1957), 291–296. (Russian)

The author applies Gersevanov's iterative procedure [Izv. Akad. Nauk SSSR. Otd. Tehn. Nauk **1943**, no. 7, 73–89] to obtain solutions of the complex functional equation $f(t+2ib)-f(t)=2it-2b$; $z=x+iy=f(\alpha+i\beta)$, $t=\alpha-ib$. Infinite trigonometrical series expansions are obtained for z, or x and y. The present author employs this functional equation in the study of drainages. The arbitrary constants in the Gersevanov-type solution are determined from the physically obvious boundary-conditions. Some typical cases are solved explicitly, and presented in diagrams. It seems, however, as if a good deal of additional work will have to be carried out before the exact criteria for convergence, etc., can be established for the different sets of parameters involved in these types of problems. *K. Bhagwandin* (Oslo).

Jain, S. C. Simple solutions of the partial differential equation for diffusion (or heat conduction). Proc. Roy. Soc. London. Ser. A. **243** (1957), 359–374.

The paper shows how approximate simple solutions may be found for linear diffusion characteristics of finite solids of various shapes. The method relates to the simple solutions available for semi-infinite slabs and it is shown how the latter may be used under a rather wide range of conditions. The simplified method appears to depend largely on a slight relaxation of the boundary conditions. It has the advantages of giving better physical insight into the problem, less laborious handling over the important diffusion range, and an ability to handle problems not amenable to the more complex series treatment. The method is illustrated by treatment of a rectangular block, a cylinder and a sphere. Estimated comparison accuracies for the finite slab show from 1 to 4% errors depending on the location in the slab and the diffusion time. General applicability is illustrated by showing how the method

may be applied to the finite slab in which the diffusion material is uniformly generated throughout.

M. G. Scherberg (Dayton, Ohio).

DiPrima, R. C. On the diffusion of tides into permeable rock of finite depth. Quart. Appl. Math. **15** (1958), 329–339.

This paper continues the work of Carrier and Munk [Proc. Symposia Appl. Math., v. 5, McGraw-Hill, New York, 1954, pp. 89–96; MR **16**, 673]. The author is now concerned with a channel of finite depth and the resulting modifications over the original work of Carrier and Munk. In many respects, the mathematical methods parallel the work of the reviewer [Amer. J. Math. **70** (1948), 730–748; MR **10**, 490]. Shallow water theory is also discussed and numerical results are presented. *A. E. Heins.*

Parsons, D. H. One-dimensional diffusion with the diffusion coefficient a linear function of concentration: reduction to an equation of the first order. Quart. Appl. Math. **15** (1957), 298–303.

The concentration $c(y)$, $y=x/t^{\frac{1}{2}}$, $-\infty<y<\infty$, is described as the solution of (DE) $2(Dc')'+yc'=0$; (BC) $c(-\infty)=c_1$, $c(+\infty)=c_2$; where $D=\alpha+\beta c$, and $0<D(c_1)<D(c_2)$. By a change of variables the problem is reduced to the consideration of (DE) $p'+up+2(1+u^2)p^2+u(3+u^2)p^3=0$ for $p(u)$ on $(-\infty, \infty)$ with the subsidiary condition (*) $\exp(-2\int_{-\infty}^{\infty} p(t)dt)=b=D(c_1)/D(c_2)$. It is shown that the latter (DE) with the (IC) $p(0)=\lambda>0$ has a solution $p(u, \lambda)$ which is positive, increasing with respect to λ, and integrable on $(-\infty, \infty)$, provided $\lambda\leq H$. Thus the equation (*) for λ has a unique solution for b sufficiently close to unity. Because of this restriction the problem cannot be considered as solved.

The proposed numerical scheme calls for the determination of b when λ is given and is therefore inefficient for solving the problem when $D(c)$ is prescribed.

I. I. Kolodner (Albuquerque, N.Mex.).

Kalašnikov, A. S. On the first boundary value problem for equations expressing one-dimensional unsteady percolation. Dokl. Akad. Nauk SSSR (N.S.) **115** (1957), 858–861. (Russian)

See also: Numerical Methods: Allred and Newhouse. **Mechanics of Particles and Systems:** Sedov. **Astronomy:** Prendergast. **Geophysics:** Duhin and Deryagin.

Optics, Electromagnetic Theory, Circuits

Thompson, B. J.; and Wolf, E. Two-beam interference with partially coherent light. J. Opt. Soc. Amer. **47** (1957), 895–902.

The authors investigate the interference pattern obtained by two beams with partially coherent light. A finite light source illuminates two small openings P_1 and P_2 in a screen and the light from the two points is investigated at a reference point.

The theory of the problem which defines the partial coherence of P_1 and P_2 is developed, and a diffractometer is used which gives Fraunhofer fringes that vary with the degree of coherence of the two points. The authors give pictures comparing the theoretically calculated pattern with photographs of the fringes, which show excellent agreement.

The list of references omits a pertinent paper by E. Schrödinger [Ann. Physik. (4) **61** (1920), 69–86] entitled „Über die Kohärenz in weitgeöffneten Bündeln".

M. Herzberger (Rochester, N.Y.).

Čekin, B. S. On change of "form" of a wave in reflection and refraction. Izv. Akad. Nauk SSSR. Ser. Geofiz. **1957**, 449–457. (Russian)

The author studies the refraction and reflection of longitudinal plane waves at two parallel planar surfaces which separate space into three homogeneous media. From known solutions of the wave equation, the general form of the refracted or reflected wave is easily found. Various special cases are studied by means of Fourier analysis, and also by letting the separation of the planar surfaces approach zero. *W. Kaplan* (Ann Arbor, Mich.).

Mints, M. Ia. Force fluctuations in an electron gas. Soviet Physics. JETP **5** (1957), 319.

Bechert, Karl. Bemerkungen zur nichtlinearen Elektrodynamik. Ann. Physik (6) **16** (1955), 97–110.

Solutions are found to the equations of non-linear electrodynamics for steady, rotating charge distributions, previously described by the author [Ann. Physik (6) **7** (1950), 369–409; **10** (1952), 430–448; MR **13**, 408; **14**, 436]. These equations are related to the Poisson equation; a first approximation corresponds to a point charge at $r=0$ and a rotating charge distribution in the vicinity of $r=a$, where a is a characteristic dimension of the particle under consideration. If the charge of the particle is given (e.g., through quantization), the total spin can be computed and is related to the fine structure constant. The author states that higher approximations to the solution remove some of the inadmissible physical properties which are implied by the first approximation.

I. Stakgold (Washington, D.C.).

★ **Delavault, Huguette. Application de la transformation de Laplace et de la transformation de Hankel à la détermination de solutions de l'équation de la chaleur et des équations de Maxwell en coordonnées cylindriques. Preface de H. Villat.** Publ. Sci. Tech. Ministère de l'Air, Notes Tech. no. **71**, Paris, 1957. v+99 pp. 1500 francs.

The second part of this short monograph is a fuller account of the work done on the use of integral transforms in solving boundary value problems by Mlle. Delavault [C. R. Acad. Sci. Paris **236** (1953), 2484–2486; **237** (1953), 1067–1068; **244** (1957), 1146–1149; MR **14**, 1090; **15**, 323; **19**, 210]. The first part (pp. 7–54) contains an account of those properties of integral transforms which are used in the second part; these are the principal properties of Laplace transforms (in one and two variables), Hankel and finite Hankel transforms and multiple Fourier transforms.

In the first chapter of the second part of the monograph the author considers the solution of the heat conduction equation $\nabla^2 F=\dot{F}$ for the infinite plate $0\leq z\leq Z$ when the initial value of F is known and the values of F on the boundaries $z=0$, $z=Z$ are prescribed. When the given functions are symmetric about $z=0$, a combination of a Laplace and a Hankel transform is used; when no such symmetry exists, the combination consists of a Laplace and a double Fourier transform. The second (and final) chapter of the second part discusses the solution of Maxwell's equations in cylindrical coordinates ρ, ϕ, z for

the region $0\leq\rho\leq R$, $z\geq 0$. The homogeneous equations are solved for the three cases: (a) zero field at $t=0$, tangential component of electric field E_t known on $\rho=R$ and on $z=0$; (b) field known at $t=0$, E_t known on $\rho=R$ and null on $z=0$; (c) zero field at $t=0$, E_t and H_t known on $z=0$. The inhomogeneous equations are solved for the case of a zero field at $t=0$, with E_t prescribed on $\rho=R$ and $z=0$. The method of solution consists of the use of a double Laplace transform in z and t, a finite Hankel transform in ρ and a finite Fourier transform in ϕ. Formal solutions are derived; the evaluation of the integrals obtained is not discussed. *I. N. Sneddon* (Glasgow).

Ichikawa, Yoshi H. Theory of collective oscillation of electrons in solids. Progr. Theoret. Phys. **18** (1957), 247–263.

Die bekannte Methode zur Berechnung der Plasmaschwingungen von D. Pines und D. Bohm [Phys. Rev. (2) **85** (1952), 338–353], nach der man das verwickelte Vielkörperproblem der gegenseitig aufeinander einwirkenden Ladungen in einem kollektiven und in einem noch übrigbleibenden individuellen Teil zerlegt, wird in der vorliegenden Arbeit auf feste Körper erweitert. Der wesentliche Unterschied gegenüber den in Gasen auftretenden Verhältnissen ist hierbei erstens, dass sich die Elektronen in einem periodischen Feld bewegen, was durch Einführung von Blochschen Eigenfunktionen berücksichtigt wird und zweitens, dass nicht mehr die Boltzmannstatistik benutzt werden kann, weil ja im festen Körper ein entartetes Elektronengas vorliegt. Den Ausgangspunkt der Betrachtungen bildet die Hamiltonsche Funktion $H=H_0+H_1$, wo

$$(1)\qquad H_0=\int \psi^*(x)\Big\{-\frac{\hbar^2}{8\pi^2 m}\Delta+U(x)\Big\}\psi(x)d^3x$$

und

$$(2)\qquad H_1=\tfrac{1}{2}\iint \psi^*(x)\psi^*(x')V(x,x')\psi(x')\psi(x)d^3xd^3x'$$

ist. $U(x)$ bedeutet das erwähnte periodische Potential und $V(x, x')$ die Coulombsche Wechselwirkung der Elektronen. Um der Fermistatistik zu genügen, werden die Eigenfunktionen ψ gewissen Vertauschungsrelationen (zweite Quantelung) unterworfen. Weiter gelingt es dann, durch Einführung der Dichtefunktion endlich die Normalkoordinaten der kollektiven Schwingungsbewegung und auch die dazu gehörenden konjugierten Impulse zu erhalten. Auf diesem Wege folgt dann die Schwingungsgleichung und daraus die Frequenz dieser Schwingungen. Mit Hilfe der neuen Veränderlichen kann die Hamiltonsche Funktion in eine solche des kollektiven Feldes, in eine welche die individuellen Bewegungen der Elektronen in einem modifizierten Felde beschreibt und endlich in eine welche die Wechselwirkungen des kollektiven Feldes und des Elektronenfeldes angibt, zerlegt werden. Im letzten Teil werden dann die auftretenden Schwingungsfrequenzen für den Fall von einem teilweise leeren und einem voll besetzten Valenzband explizit angegeben. Der Vergleich mit der Erfahrung bleibt einer weiteren Veröffentlichung vorbehalten. *T. Neugebauer.*

Good, R. H., Jr. Electromagnetic multipole radiation. Ann. Physics **1** (1957), 213–220.

In an earlier paper [Good, Phys. Rev. (2) **105** (1957), 1914–1919; MR **19**, 216] the author expressed the electromagnetic field equations in a form emphasizing their interpretation in terms of particles (photons). This analysis is extended in the present paper to a discussion of multipole radiation. From a mathematical point of view this is accomplished by expanding the field vectors in series of vector spherical harmonics. The particle interpretation is contained in the statement that these functions provide a basis of eigenfunctions for the angular momentum operators of the field. The analysis is an extension of that given by Blatt and Weisskopf [Theoretical nuclear physics, Wiley, New York, 1952, ch. 12].

E. L. Hill (Minneapolis, Minn.).

Ušerovič, I. S. On approximate solution of the transport equation for radiative energy. Zaporiz. Derž. Ped. Inst. Nauk. Zap. Fiz.-Mat. Ser. **2** (1956), 3–15. (Ukrainian)

Vaĭnštein, B. K. On the theory of a method of radial scattering. Akad. Nauk SSSR. Kristallografiya **2** (1957), 29–37. (Russian)

The radial distribution function which occurs in scattering of electrons or X-rays by amorphous substances is studied theoretically. A new method of normalisation has been put forward which allows one to utilise the whole scattering intensity curve and not only the outer part. The dependence of the peak heights on the atomic number has been clarified and expressions have been derived for the scattering of electrons and of X-rays.

V. Vand (University Park. Pa.).

Grinberg, G. A. A new method for solution of a problem of diffraction of electromagnetic waves on a surface using the unlimited rectilinear slit adopted in related problems. Ž. Tehn. Fiz. **27** (1957), 2595–2605. (Russian)

The author considers diffraction by a perfectly conducting screen, lying in the xz-plane, in which there is an infinitely long slit parallel to the z-axis. The integral $(\pi k/c)\int_{(S)} j(\xi)H_0^{(2)}(k|x-\xi|)d\xi=E^0(x,0)$ expresses the vanishing of the total electric field at the surface of the screen. Here (S) is the portion of the x-axis lying in the screen; $j(\xi)$, the current density, is the sum of $j_1(\xi)$, the density on $y=+0$, and $j_2(\xi)$, that on $y=-0$; and $j_2(\xi)-j_1(\xi)=(c/2\pi)H^0{}_x(\xi,0)$. For the problem of reflection of the same external electromagnetic field $E^0(x,y)$, $H^0(x,y)$ from a perfectly conducting screen occupying the entire plane $y=0$, one obtains an integral which is formally the same except that the limits are $\pm\infty$ and $j(\xi)$ is replaced by $\bar{j}(\xi)$, the density of the induced current. Since the right members are the same, the two equations can be combined, and as a result one obtains

$$(\pi k/c)\int_{(S)} j_2(\xi)H_0^{(2)}(k|x-\xi|)d\xi=\bar{E}(x),$$

where $\bar{E}(x)$ is interpreted as the electric field induced at x by a current with density $(c/4\pi)H^0{}_x(\xi,0)$ on the complement of (S) on the x-axis. The latter problem is called the "key".

The method is applied to the Sommerfeld problem (diffraction by a perfectly conducting half-plane), the "key" to which is the problem of finding currents induced in the half-plane by a line of current sources lying in the complementary halfplane, parallel to the edge and an arbitrary distance from it. The author then treats the problem of diffraction by an unbounded plane with rectilinear slit of an arbitrary external field, obtaining an asymptotic solution for wavelength small relative to the

width of the slit. For a plane wave at normal incidence, the approximation is good even when the wavelength equals the width. *R. N. Goss* (San Diego, Calif.).

Ivanenko, I. P. Equilibrium spectrum of electrons and photons with account of scattering. Soviet Physics. JETP **5** (1957), 204–208.

An exact solution of the equation for the equilibrium spectrum of electrons and photons has been found with account of scattering, i.e., an expression is obtained for the angular and energetic distributions of particles at the shower maximum in heavy elements. *Author's summary.*

Cerenkov, N. A. Effect of a longitudinal magnetic field on the multiple scattering of particles. Soviet Physics. JETP **5** (1957), 320–321.

Bremmer, H. Asymptotic developments and scattering theory in terms of a vector combining the electric and magnetic fields. I.R.E. Trans. **AP-4** (1956), 264–265.

Northover, F. H. Radiation and surface currents from a slot on an infinite conducting cylinder. Canad. J. Phys. **36** (1958), 206–217.

In this paper the author discusses the approximations involved in the Papas procedure for the determination of the far-zone field and the radiation conductance of a radiating slot on a perfectly conducting cylinder of infinite length. *C. H. Papas* (Pasadena, Calif.).

Takahashi, I.; and Watanabe, T. Transient phenomena in the wave guide. Mem. Coll. Sci. Univ. Kyoto. Ser. A. **28** (1957), 221–230.

The authors apply a result presented by Stratton [Electromagnetic theory, McGraw-Hill, New York, 1941, Sec. 5.13] for plane wave transients to the solution of transient problems in perfectly conducting waveguides. Through the standard use of time-dependent Hertz potential functions, formal mode expansions of the conventional type are obtained for the case of axially directed electric or magnetic current elements, with the current impressed at time $t=0$ having the form of a step function or of a rectangular pulse. Approximate formulas are presented for the calculation of the field in each mode for small values of $t'=t-z/c$, where z is the axial distance from the source and c is the velocity of light in the medium. *L. B. Felsen* (Brooklyn, N.Y.).

Papoulis, A. Limits on the zeros of a network determinant. Quart. Appl. Math. **15** (1957), 193–194.

From the matrix of a network, consisting of *R-C* or *L-C* elements, control sources and feedback, limits are given on the location of the roots. *G. Kron.*

Gaponov, A. V. The method of imposing idealized connections in the general theory of electrical machines. Trudy Gor'kov. Issled. Fiz.-Tehn. Inst. Radiofiz. Fak GGU. Uč. Zap. **30** (1956), 142–158. (Russian)

The types of reference frames that can be assumed on an idealized rotating electrical machine are discussed. *G. Kron* (Schenectady, N.Y.).

See also: Classical Thermodynamics, Heat Transfer: Li and Ting. **Quantum Mechanics:** Schmutzer; Vaghi. **Relativity:** Mariot; Pham; Takeno and Ueno; Lanczos.

Classical Thermodynamics, Heat Transfer

Li, James C. M.; and Ting, Tsuan Wu. Thermodynamics for elastic solids in the electrostatic field. I. General formulation. J. Chem. Phys. **27** (1957), 693–700.

An important chapter in the phenomenological theory of thermodynamics is devoted to the derivation of the relations which exist among the various measured coefficients (elastic, piezoelectric, and so on). In a mathematical sense these relations simply are necessary connecting equations among the various partial derivatives of the thermodynamic functions. For anisotropic bodies the number of such relations is so large that considerable effort is required to organize them in a systematic manner. The authors have undertaken this task for the case of a general anisotropic body in an external electrostatic field. Systematic tables are given, corresponding to various choices of the independent variables. *E. L. Hill.*

Manfredi, Bianca. Soluzioni numeriche in problemi di flusso lineare di calore. Riv. Mat. Univ. Parma **6** (1955), 141–155.

Quantum Mechanics

Halpern, Francis R. Method of moments in quantum mechanics. Phys. Rev. (2) **107** (1957), 1145–1147.

An approximate method is described for finding the eigenvalues and eigenfunctions of the Hamiltonian of a quantum-mechanical problem. It is based on the method of moments employed in probability theory, and the theory of orthogonal polynomials. The advantages of the method are that it is independent of the magnitude of the interaction and that the basic quantities are relatively simple to compute. *C. Froese* (Vancouver, B.C.).

Takahashi, Y. On the generalized Ward identity. Nuovo Cimento (10) **6** (1957), 371–375.

A formal proof is given of Ward's identity without making use of perturbation expansions. Renormalized quantities are used exclusively, and the identity is shown to be a consequence of the gauge invariance of the theory (and of charge conservation) as had been conjectured. A number of other identities are also derived. *F. Rohrlich* (Iowa City, Iowa).

Mušicki, Đ. L'application du principe de Pfaff en mécanique quantique. Bull. Acad. Serbe Sci. Cl. Sci. Math. Nat. (N.S.) **10** (1956), no. 2, 25–28.

A. D. Bilimovitch [Glas Srpske Akad. Nauka **189** (1946), 121–152; MR **11**, 218] formulated the so-called Pfaff's general principle of mechanics from which it is possible, as shown by himself and Angelitch (Anđelić), to deduce various differential equations of motion in mechanics. These equations are Pfaff's equations attached to a particular Pfaffian form which is a specially modified element of action in Hamilton's sense. In this paper it is shown that it is possible to deduce from the Pfaff's principle, after some generalizations, the Schrödinger's equation for stationary states also, starting from the kinetic and potential energy of the waves of matter. *T. P. Anđelić* (Belgrade).

Mušicki, Đ. L'équation relativiste des ondes de la matière et le principe de Pfaff. Bull. Acad. Serbe Sci. Cl. Sci. Math. Nat. (N.S.) **10** (1956), no. 2, 29–31.

The author demonstrates here how it is possible (cf. the

above review) to deduce from Pfaff's principle the Klein-Gordon equation for waves of matter. *T. P. Anđelić.*

Zumino, Bruno. On the formal theory of collision and reaction processes. Div. Electromag. Res., Inst. Math. Sci., New York Univ., Res. Rep. No. **CX-23** (1956), i+31 pp.

A formal theory is given for particle reactions such as scattering, absorption, or emission. In particular, the time-dependent Schroedinger equation is solved for a] perturbation of a system which in an unperturbed state has a mixed eigenspectrum. The present paper improves upon the existing treatments by the use of the resolvent operator, which gives an unambiguous method of evaluating the successive approximations. In the past the resolvent method has not been applied in full generality to continuous spectra [e.g. W. Heitler, The quantum theory of radiation, 3rd ed., Oxford, 1954]. First formulae for the resolvent operator of a perturbed Hamiltonian H are given. The resolvent of H is

$$G(a)=(a-H)^{-1},$$

where a is a complex parameter. It is just this function which enters as the kernel in the Cauchy formula for an analytic function of H. Applying this formalism, the author discusses first discrete spectra, and then mixed spectra. Finally the asymptotic behavior of the solution of the time dependent Schroedinger problem is discussed, and two asymptotic formulae are given. One is for the limit of infinite time, the other for large times and weak interactions. *M. J. Moravcsik* (Livermore, Calif.).

Brussaard, P. J.; and Tolhoek, H. A. Classical limits of Clebsch-Gordan coefficients, Racah coefficients and $D^{l}_{mn}(\varphi, \vartheta, \psi)$-functions. Physica **23** (1957), 955–971.

This nice paper deals with the classical limits of functions used in adding angular momenta in quantum mechanics. The functions are the Clebsch-Gordan coefficients [see, e.g., E. U. Condon and G. H. Shortley, The theory of atomic spectra, Cambridge, 1935], the Racah coefficients [Phys. Rev. (2) **62** (1942), 438–462], and the Wigner D-functions [Gruppentheorie und ihre Anwendung auf die Quantenmechanik der Atomspektren, Vieweg, Braunschweig 1931]. The classical limit means that some or all of the angular momenta are taken to be very large. In this case the usual quantum mechanical probability interpretation of the square of the above coefficients can be carried over into classical probabilities. From a mathematical point of view, several new asymptotic formulae are derived for the functions in question, for large angular momenta. For the D functions WKB-expressions are also derived and their relation to the classical limit is discussed. In general, an averaging over the rapidly oscillating functions give the classical analogue. The Legendre polynomials are used for illustrating some of the points. *M. J. Moravcsik.*

★**Atkin, R. H. Mathematics and wave mechanics.** John Wiley & Sons, New York, N.Y. 1957. xv+348 pp. $6.00.

This book is intended for students, with a training predominantly in physics or chemistry, who require sufficient mathematics to follow the formal details of quantum mechanics. Ranging as it does from introductory chapters on the calculus and vector and matrix algebra, through such parts of classical physics as mechanics and electromagnetism, to topics in quantum chemistry, statistical mechanics and the theory of the quantised field, it must necessarily be condensed and to some degree superficial. For example, the time dependent Schrödinger equation is introduced twice in Chapter 10; in the correct form and also in the original incorrect form of Schrödinger's early papers [Abhandlungen zur Wellenmechanik, 2nd ed., Barth, Leipzig, 1928], but no explanatory comment is given. However, the logical arrangement of the material is commendable, and if intelligently employed by a teacher, it covers a course suited to those who do not intend to concern themselves with mathematical and physical complexities.

The printing is good although there are a fair number of minor misprints. An ample supply of practice exercises are included with each chapter. The chapter headings are: Analysis (I); analysis (II); vectors, determinants, and matrices; surface and multiple integrals; differential equations; classical mechanics; vector field theory and its application; wave motion and the electromagnetic field; the new quantum mechanics; the wave equation; perturbation theory; the spin of the electron; quantum chemistry; the new statistics; the theory of the quantised field. *C. A. Hurst* (Adelaide).

Sokoloff, J.; and Hamermesh, M. Calculation of scattering from a complex square well by a variational method. Ann. Physics **2** (1957), 157–165.

In the course of numerical calculations of scattering from a complex square well, we found it useful to obtain estimates of phase shifts by quick hand computation. This can be accomplished by a simple variational method. The phase shifts for all angular momenta can be obtained in closed form convenient for computation.

Authors' summary.

Harmuth, Henning. Die Unschärferelation in den Dirac-Gleichungen und in der relativistischen Schrödinger-Gleichung. Z. Naturf. **11a** (1956), 101–118.

Tredgold, R. H. Density matrix and the many-body problem. Phys. Rev. (2) **105** (1957), 1421–1423.

The application of the two-particle density matrix to the calculation of approximate ground states for many-body systems is discussed. It is shown that the subsidiary conditions employed by Mayer in this context are insufficient to ensure that a valid result is obtained. The formulation of adequate subsidiary conditions presents considerable difficulty. (Author's abstract.)

P.-O. Löwdin (Uppsala).

Chew, G. F.; Goldberger, M. L.; Low, F. E.; and Nambu, Y. Application of dispersion relations to low-energy meson-nucleon scattering. Phys. Rev. (2) **106** (1957), 1337–1344.

Relativistic dispersion relations are used to derive equations for low-energy S-, P-, and D-wave meson-nucleon scattering under the assumption that the (3,3) resonance dominates the dispersion integrals. The P-wave equations so obtained differ only slightly from those of the static fixed-source theory. The conclusions of the static theory are re-examined in the light of their new derivation. (Author's abstract.) *P.-O. Löwdin.*

McWeeny, R. The density matrix in self-consistent field theory. II. Applications in the molecular orbital theory of conjugated systems. Proc. Roy. Soc. London Ser. A. **237** (1956), 355–371.

The methods of a previous paper [same Proc. **235**

(1956), 496–509; MR **18**, 443] are applied in the theory of conjugated systems. The density matrix — which in this case is the array of "charges" and "bond orders" — may be calculated for a general π-electron system, in self-consistent field (s.c.f.) approximation, by iterative refinement of an initial estimate. In choosing an initial estimate it is possible to make use of known Hückel theory solutions for simple hydrocarbons, building up approximations for the more complicated systems by changing parameters (describing, for example, the insertion of hetero-atoms of the inter-connexion of different fragments). This may be done by a density matrix perturbation method which avoids all reference to the wave function itself. In this way, Hückel approximations for rather general systems can be written down with very little calculation and, at the same time, there are interesting chemical applications along the lines of Coulson and Longuet-Higgins. Finally, self-consistency can be introduced, and the perturbation method can be carried over into the s.c.f. theory; this permits a rather more realistic discussion of chemical properties and explains the success, in certain cases, of the simple Hückel approach. (Author's abstract.) *P.-O. Löwdin.*

McWeeny, R. The density matrix in self-consistent field theory. III. Generalizations of the theory. Proc. Roy. Soc. London. Ser. A. **241** (1957), 239–256.

The self-consistent field theory of a previous paper [same Proc. **235** (1956), 496–509; MR **18**, 443] is generalized so as to apply to a system in which some orbitals are doubly occupied and others singly. Starting from an initial approximation, the density matrices for both the closed shell and the "open" shell are simultaneously improved by an iterative method until self-consistency is achieved.

A second generalization deals with the case in which several shells can profitably be distinguished, not by a difference of orbital occupation numbers but by virtue of their weak coupling (e.g. the atomic K, L, M shells of argon). In this case it is best to consider each shell individually (their number and type now being unrestricted), achieving approximate self-consistency in one shell at a time.

The nature of the solutions is discussed. In cases of high symmetry, degeneracy occurs and a single determinant is not an acceptable state function; at the same time the self-consistent solution has awkward symmetry properties. This latter difficulty can be avoided with only slight loss of accuracy by applying suitable constraints. The wider use of constraints, as a means of facilitating solution, is discussed. Finally, it is shown how, having obtained self-consistent density matrices, the orbitals themselves may be extracted in readiness for a configuration interaction calculation; the degenerate case, in which interaction is essential, is briefly discussed. (Author's abstract.) *P.-O. Löwdin* (Uppsala).

Blohincev, D. I. The non-linear field theory and the theory of relativity. Nuovo Cimento (10) **4** (1956), supplemento, 629–634.

Non-linear relativistic field theories are considered in a rather general way and it is shown that they cannot remove the divergence difficulties and remain consistent with the causality condition. *P. T. Matthews.*

Novozhilov, Iu. V. Quantum field theory with causal operators. Soviet Physics. JETP **4** (1957), 553–561.

A mathematical formalism is set up for the space-time description of field theory, starting from an action principle expressed in terms of the, so-called, causal operators introduced by F. Coester [Phys. Rev. (2) **95** (1954), 1318–1323; MR **16**, 320]. The variables canonical to the causal operators, in a generalised sense, play the role of external sources and the equations following from the action principle are identical with the functional differential equations of Schwinger [Proc. Nat. Acad. Sci. U.S.A. **37** (1951), 452–455, 455–459; MR **13**, 520].

P. T. Matthews (London).

Lieb, E. H. A non-perturbation method for non-linear field theories. Proc. Roy. Soc. London. Ser. A. **241** (1957), 339–363.

The non-linear interaction of a scalar field ϕ, through a term $\lambda\phi^4$ in the Lagrangian, is quantised, à la Feynman, by functional integration. The field energy in the presence of an external source, and the one- and two-meson propagators are expressed as functional integrals over the field ϕ. [P. T. Matthews and A. Salam, Nuovo Cimento (10) **2** (1955), 120–134; MR **17**, 693]. These are evaluated approximately by a method proposed by Edwards [S. F. Edwards, Proc. Roy. Soc. London. Ser. A. **232** (1955), 371–376, 377–389; MR **17**, 927], which replaces the quartic exponent by an equivalent quadratic, and treats the difference as a perturbation. The results are renormalized and agree with conventional perturbation theory in the limit of small λ. For strong coupling the solutions give a physically reasonable account of the nonlinear features of the problem. However, for a strong external source, the field energy may become complex and an unphysical bound state (whose binding energy increases as the coupling strength decreases) appears in the two-particle propagator. It is not clear whether these difficulties are a product of the approximations, or are inherent in the theory.

The paper tackles difficult problems by refreshingly original methods. *P. T. Matthews* (London).

Rzewuski, J. Some remarks on non-local theories. Bull. Acad. Polon. Sci. Cl. III. **5** (1957), 393–400, XXXII. (Russian summary)

Equations for bilocal fields with non-local interactions are transformed into equations for local fields with non-local interactions. The possibility of obtaining both a mass spectrum and convergence factors within this general framework is discussed. *P. T. Matthews.*

Bollini, C. G. On the quantization of tensor fields with zero mass. Nuovo Cimento (10) **6** (1957), 1034–1039.

"A general formulation of quantum field theory for massless tensor fields, compatible with the supplementary condition, is given. The free field (or the interacting field in the interaction representation) is expressed in terms of its components "along" a certain transversal polarization tensor. These components are taken as canonical coordinates, on which the canonical commutation relations are imposed. The corresponding commutation relations of the field components are given." (From the author's summary.) *G. Källén* (Copenhagen).

Nambu, Y. Parametric representations of general Green's functions. Nuovo Cimento (10) **6** (1957), 1064–1083.

The "general Green's functions" are defined to be the vacuum expectation values of time ordered products of field operators. Special examples of these functions are the twofold expectation value $\langle 0|T(\psi(x)\bar{\psi}(y))|0\rangle$ (the

"electron Green's function"), the threefold expectation value $\langle 0|T(\psi(x)\bar{\psi}(y)A(z))|0\rangle$ (the "vertex part") etc. with the electron field $\psi(x)$ and the electromagnetic field $A(x)$ of quantum electrodynamics. By an examination of every order of the perturbation theory expressions in conventional quantum electrodynamics the author obtains the result that the Fourier transform $G(k_1, \cdots, k_n)$ of such a function can be written in the form

$$G(k)=\int\cdots\int[\Sigma\,\beta_\mu l_\mu{}^2+M^2-ie]^{-n}\sigma(M^2,\beta_\mu)dM^2\prod d\beta_\mu,$$

where the l_μ's are certain linear combinations of the vectors k_μ and where the variables β_μ and M^2 are integrated over positive values only. If the $l_\mu{}^2$'s are considered as independent complex numbers, this formula exhibits $G(k)$ as the boundary value of an analytic function of the $l_\mu{}^2$'s regular, e.g., when all $l_\mu{}^2$'s have the same signs of their imaginary parts. It can be shown with the aid of very general considerations (Lorentz invariants, reasonable mass spectrum, local commutativity) that $G(k)$ is always the boundary value of an analytic function of the $l_\mu{}^2$'s, but it can also be shown that these arguments will not, in general, yield a regularity domain of these functions as large as indicated by the formula displayed above. Therefore, this result of the author raises the interesting question whether there are further general properties of quantized field theories not explored so far that will enlarge the analyticity domains of the functions $G(k)$, or whether the result of the author is a consequence of perturbation theory for the very particular interaction one has in conventional quantum electrodynamics. *G. Källén.*

Solov'ev, V. G. Investigation of a model in quantum field theory. Soviet Physics. JETP **5** (1957), 859–866.

The model of this paper consists of a pseudoscalar neutral meson field in interaction with nucleons. The author introduces two "limiting momenta" λ_φ for the meson field and λ_ψ for the nucleon field and uses them as cut-offs in the theory. To make the model mathematically manageable, the author puts $\lambda_\varphi=0$ and lets λ_ψ tend to infinity afterwards. This means that only constant meson fields have an interaction with the point nucleons of the model. Because of this simplification, the functional integral representation of the Green's function of the nucleons reduces to an ordinary integral over the strengths of the constant meson field. As the classical Green's function of a nucleon in a constant external field can be computed explicitly, the author can also get an exact expression for the corresponding function in the quantized theory. He then gives a rather detailed discussion of the renormalizations necessary in the model. Further, he shows that the Green's function emerging at the end fulfills all the conventional criteriae that follow from a positive definite metric in the Hilbert space. This indicates that the model discussed here does not have the difficulties with "negative probabilities" encountered, e.g., in the so called Lee model [T. D. Lee, Phys. Rev. (2) **95** (1954), 1329–1334; MR **16**, 316; G. Källén and W. Pauli, Danske Vid. Selsk. Mat.-Fys. Medd. **30** (1955), no. 7; MR **17**, 927]. *G. Källén* (Copenhagen).

★**Sokolow, A. Quantenelektrodynamik.** Akademie-Verlag, Berlin, 1957. x+324 pp. DM 29.00.

A translation into German by Dieter Bebel and Ulrich Tarnick of the first half, the part by A. Sokolov, of the Russian book reviewed in MR **14**, 1044. A translation of the part by D. Ivanenko is promised. In the present part the author has made several additions, particularly in sections 31 (Multipolstrahlung), 36 and following (Theorie der Streuung unter Berücksichtigung der Strahlungsdämpfung) and 48 and following (Theorie des Positroniumatoms).

Kimura, Toshiei. On the quantum field theory of the interaction between the graviton and the matter field. Progr. Theoret. Phys. **16** (1956), 157–176.

The author investigates the quantum dynamical interactions between particles of spin 0, $\frac{1}{2}$ and 1 and the gravitational field. For the most part, only a linearized gravitational Lagrangian is used, subject to the DeDonder coordinate conditions

$$\frac{\partial}{\partial x^\mu}\,g_{\mu\nu}=0,\quad g^{\mu\nu}=g^{\mu\nu}[-\det(g_{\alpha\beta})]^{\frac{1}{2}}.$$

Thus the whole question as to whether the quantization procedure is invariant under general coordinate transformations is not discussed. Within this framework, the interaction representation is set up. One of the results obtained is that the potential between two particles interacting gravitationally is the same, up to the second order in the gravitational field, as that obtained by Einstein, Infeld and Hoffmann [Ann. of Math. (2) **39** (1938), 65–100] classically, if one neglects radiative corrections. The discussion of the divergence problem does not seem to correctly take into account the general theorem that the singularity of the propagators cannot be softened by radiative corrections for couplings of the conventional type [for comments on this question, see the discussion by S. Deser in the proceedings of the conference on General Relativity held at Chapel Hill, North Carolina, January 1956 (to be published)]. *R. Arnowitt.*

Kimura, Toshiei. On the quantum field theory of the interaction between the graviton and the matter field. II. Formulation without the coordinate condition. Progr. Theoret. Phys. **16** (1956), 555–568.

In the paper reviewed above, the linearized gravitational Lagrangian subject to the DeDonder coordinate condition is quantized following the methods of Yang and Feldman [Phys. Rev. (2) **79** (1950), 972–978; MR **12**, 569]. This paper still deals with the linearized Lagrangian but does not invoke a particular coordinate condition. The problem of the constraint equations is dealt with by the quantization procedures of Dirac [Canad. J. Math. **2** (1950), 129–148; MR **13**, 306]. In this technique, constraints are divided into two classes depending on whether or not a constraint's Poisson bracket with every other constraint vanishes. Quantization can then be carried out in a consistent fashion by modifying the definition of the Poisson bracket. When one considers the problem of a spinor field interacting with the gravitational field, this modification produces the following interesting result: the spinor creation operators involve the simultaneous creation of the spinor particle and the gravitational field surrounding it.

The quantization scheme followed here is shown to be equivalent to the Yang-Feldman method of the previous paper. *R. Arnowitt* (Syracuse, N.Y.).

Kibble, T. W. B.; and Polkinghorne, J. C. On Schwinger's variational principle. Proc. Roy. Soc. London. Ser. A. **243** (1957), 252–263.

This paper deals with apparently arbitrary features of a variational method elaborated by Schwinger [Phys. Rev. (2) **91** (1953), 713–728; MR **15**, 81] for field-theoretical

problems. The Lagrangian function is expressed in a canonical form linear in the generalized velocities. Infinitesimal canonical transformations are considered and it is found that they admit two different kinds of variation of the co-ordinates, independent of and linear in the coordinates, respectively. The first kind allows field quantization only in accordance with Bose or Fermi statistics. The second kind allows, in addition, quantization in accordance with more general statistics. The additional postulates required by Schwinger's method are made precise. *H. S. Green* (Adelaide).

Schmutzer, Ernst. Minkowski-Elektrodynamik als Ergebnis einer feldtheoretischen Untersuchung. Ann. Physik (6) **20** (1957), 349–354.

A historical question is raised and answered: whether the energy-momentum tensor of the electromagnetic field in interaction is of the Minkowski or the Abraham type (the latter containing the four-velocities explicitly). Starting with a Lorentz and gauge-invariant Lagrangian, the symmetrized energy-momentum tensor is found by standard methods and the only separation into "field" and "matter" tensors which preserves all invariance properties is seen to yield the Minkowski tensor, in agreement with correspondence arguments, — and with no surprise to workers in present-day field theory.
F. Rohrlich (Iowa City, Iowa).

Barashenkov, V. S. Theory of nonlocal interaction. Soviet Physics. JETP **4** (1957), 709–712.

The possibility of a Hamiltonian structure in the theory of nonlocal interaction is considered. By way of example, the one-dimensional oscillator is considered by Pauli's method. The initial integro-differential equations of motion and the canonical system of equations of motion obtained by Pauli's method are not equivalent. The procedure of Hayashi is shown to be internally inconsistent in any given order. Thus, conclusions about the essentially non-Hamiltonian character of nonlocal interactions are confirmed. *Author's summary.*

Sucher, J. *S*-matrix formalism for level-shift calculations. Phys. Rev. (2) **107** (1957), 1448–1449.

An exact formula for the level shift is given in terms of the adiabatic *S*-matrix constructed from the perturbation which produces the shift. The formula is used to discuss the role of the subsidiary condition in quantum electrodynamic level-shift calculations. *Author's summary.*

Vaghi, Carla. Energia di campi spazio-temporali emisimmetrici. Boll. Un. Mat. Ital. (3) **12** (1957), 264–268.

The author sets out to determine the expression for the energy tensor of a generalized electromagnetic field, including that of Maxwell and that of the Yukawa meson field. The field is represented by a four-dimensional space-time tensor, composed of an irrotational tensor and a solenoidal tensor. For the calculation of the energy tensor, a method of D. Hilbert is extended to the present case. Using a variational principle, the derivation is straightforward. In the case of irrotational fields the present expression reduces to the energy tensor of Proca-Yukawa. *M. J. O. Strutt* (Zurich).

Novozhilov, Iu. V. Scale transformation and the virial theorem in quantum field theory. Soviet Physics. JETP **5** (1957), 138–140.

Grigor'ev, V. I. Quantum field theoretical solutions without perturbation theory. Soviet Physics. JETP **5** (1957), 109–111.

Koide, Shoichiro; Sekiyama, Hisao; and Nagashima, Toshio. One-center molecular orbital wave function for methane. J. Phys. Soc. Japan **12** (1957), 1016–1021.

A molecular orbital wave function for methane is expanded in terms of spherical harmonics centred on the carbon nucleus and the coefficients in this expansion are determined by the variational method. It is found that the angular dependence of the nuclear potential has quite a small effect on the total energy, but that the total electronic charge density shows significant departures from spherical symmetry. *A. C. Hurley.*

Brodersen, Svend. A simplified procedure for calculating the complete harmonic potential function of a molecule from the vibrational frequencies. Mat.-Fys. Skr. Danske Vid. Selsk. **1** (1957), no. 4, 34 pp.

It is shown that the calculation of the internal potential energy function of a molecule from the observed vibrational frequencies may be simplified by a consistent use of symmetry coordinates composed from locally cartesian coordinates. This procedure, unlike the usual method based on internal symmetry coordinates, does not involve removal of translations and relations from the symmetry coordinates. Instead, a number of relations between the force constants are introduced. Ethylene, acetylene and ethane are used to illustrate the new method.
A. C. Hurley (Melbourne).

Ishiguro, Eiichi; Kayama, Kunifusa; Mizuno, Yukio; Arai, Tadashi; and Sakamoto, Michiko. Tables useful for the calculation of the molecular integrals. X. Nat. Sci. Rep. Ochanomizu Univ. **7** (1956), 63–94.

The basic energy integrals for a homonuclear diatomic molecule involving $1s$, $2s$, $2s^0$ and $2p$ Slater functions have previously been published [E. Ishiguro, T. Arai, M. Sakamoto and K. Takayanagi, same Rep. **6** (1955), 157–181; MR **17**, 1167] for four sets of parameter values (appropriate for the molecules Li_2 and O_2). Here these integrals are transformed into integrals over orthonormal molecular orbitals formed by a Schmidt orthogonolization of the simple molecular orbitals of each symmetry class. Integrals required for the calculation of dipole moments, diamagnetic susceptibilities and transition probabilities are also given. *A. C. Hurley* (Melbourne).

Coulson, C. A.; and Kearsley, Mary J. Colour centres in irradiated diamonds. I. Proc. Roy. Soc. London. Ser. A. **241** (1957), 433–454.

The absorption spectrum of an isolated vacancy in an otherwise perfect diamond lattice is calculated by applying molecular orbital methods to a "defect molecule". If calculated values are used for all the energy integrals involved, no satisfactory interpretation can be given for the observed absorption at about 2 electron volts. Empirical modifications of some of these integrals, similar to those found necessary in calculations of hydrocarbon spectra, are then considered. The modified calculation leads to a quantitatively satisfactory interpretation of this observed absorption as due to spin and orbitally allowed electronic transitions of symmetry $'E \rightarrow {}'T_2$ in the neighbourhood of isolated neutral vacancies.
A. C. Hurley (Melbourne).

Bohr, A. Problems of nuclear structure. Nuovo Cimento (10) **3** (1956), supplemento, 1091–1120.

This lecture gives an excellent review of the physical ideas underlying the collective particle model for the description of nuclear structure. *A. Salam* (London).

Ahrens, Tino. Nuclear matrix element relations in the Fermi theory of beta decay. Progr. Theoret. Phys. **18** (1957), 331–344.

The calculation of nuclear matrix elements in beta decay has in the past been done in two ways. One [see, e.g., Ahrens and Feenberg, Phys. Rev. (2) **86** (1952), 64–68] used commutation relations and general nuclear model arguments, while the other [see, e.g., D. C. Peaslee, ibid. **91** (1953), 1447–1457] used a specific nuclear force. The present paper illustrates the extent to which all nuclear matrix element calculations must be based on results obtainable from commutation relations. Two of these relations contain terms multiplied by the nucleon mass while two do not. The results are then applied to those terms in first and second forbidden correction factors which are proportional to the square of the Coulomb barrier. *M. J. Moravcsik* (Livermore, Calif.).

Foldy, L. L. Photodisintegration of the lightest nuclei. Phys. Rev. (2) **107** (1957), 1303–1305.

It is shown that for nuclei in which the ground state wave function is symmetric in the space coordinates of all the nucleons, such as H^2, H^3, He^3, He^4, the integrated bremsstrahlung-weighted cross section for electric dipole absorption is simply related to the mean-square radius of the nucleus, independent of the existence of correlations between the motions of the nucleons.

From the author's summary.

Melikian, E. G. Internal Compton effect in pair conversion. Soviet Physics. JETP **5** (1957), 331–333.

Ivanter, I. G.; and Okun, L. B. On the theory of scattering of particles by nuclei. Soviet Physics. JETP **5** (1957), 340–341.

★ **Daudel, Raymond. Les fondements de la chimie théorique. Mécanique ondulatoire appliquée à l'étude des atomes et des molécules. Préface de M. Louis de Broglie.** Gauthier-Villars, Paris, 1956. x+236 pp. 3500 francs.

Bernstein, Jeremy. Scattering of K^+ particles from protons and deuterons. Phys. Rev. (2) **105** (1957), 1853–1858.

Schwinger [Phys. Rev. (2), **104**, 1164–1172 (1956)] has proposed a theory of K mesons and hyperons which accounts for the ability of K mesons to show positive or negative intrinsic parity in decaying into π-mesons while showing only one lifetime and one mass. In the direct $K-\pi$ coupling which he uses, the intrinsic parity is not a constant of the motion. Parity-symmetric fields are constructed corresponding to K_1, K_2 particles. An invariant coupling with the meson field can be constructed and calculations are made of scattering of K^+ particles from nucleons and deuterium. Qualitative agreement with experimental proton scattering is obtained and, theoretically, no elastic scattering from deuterium is expected.

C. Strachan (Aberdeen).

Avrorin, E. N.; and Fradkin, E. S. Renormalizability of pseudoscalar meson theory with pseudovector coupling. Soviet Physics. JETP **3** (1957), 862–865.

If the polarization of the vacuum in pseudovector coupling is calculated to second order in the coupling constant, by techniques which are not in contradiction with the "equivalence" with pseudoscalar coupling, then the result diverges only logarithmically. The modified perturbation theory proposed by Ning Hu [Phys. Rev. (2) **80** (1950), 1109–1110; MR **12**, 573] is thus no more convergent than conventional perturbation theory. This is consistent with the general results of H. Lehmann [Nuovo Cimento (9) **11** (1954), 342–357; MR **17**, 332].

P. T. Matthews (London).

Lapidus, L. I. Isotopic invariance and the creation of particles. Soviet Physics. JETP **4** (1957), 740–748.

An isotopic spin analysis is made of various processes, including nucleon-anti-nucleon pair production in nucleon-nucleon collisions, and the creation of K-meson and nucleon pairs in pion nucleon collisions.

P. T. Matthews (London).

Hamaguchi, M. On the viscous fluid model in multiple production of mesons. Nuovo Cimento (10) **5** (1957), 1622–1635.

This is a direct continuation of a previous paper by the author [Nuovo Cimento (10) **4** (1956), 1242–1261; MR **18**, 856]. The limits of the application of the perturbation method with respect to the viscosity are given. The multiplicity of particles and angular distributions in the laboratory frame are calculated, and it is shown that the viscous coefficients can be roughly estimated by comparison with experimental data. *P. T. Matthews.*

Ioffe, B. L. Dispersion relations for scattering and photoproduction. Soviet Physics. JETP **4** (1957), 534–544.

A derivation of dispersion relations for meson-nucleon scattering and photoproduction of mesons is attempted [see also M. Goldberger, Phys. Rev. (2) **99** (1955), 979–985]. The results obtained parallel those of Goldberger, Nambu, Gell-Mann, Salam, Gilbert, Takeda, Capps and others [for full references see M. Goldberger, High Energy Nuclear Physics, Proc. 6th Annual Rochester Conference, 1956, Interscience, New York, 1956, pp. I-1–I-15]. The derivation is as non-rigorous as the derivations of the authors quoted above. However it differs from theirs in not using the causality condition in the operator form $[\phi(x), \phi(y)]=0$ if $(x-y)^2<0$, but merely as the statement of the impossibility of propagation of signals with velocities exceeding that of light. This means that if an analytical function exists for the S-matrix element, which does not involve $[\phi(x), \phi(y)]$, (and the author has not been able to propose one), the dispersion relations would still remain valid. It is shown that violation of causality such that signals can reach a point r, t from all points r', t' which satisfy the condition $c^2(t-t')-(r-r')^2>-l_0^2$, where l_0 is some constant of the order of a nuclear distance, would not affect the dispersion relations. In this connection the following remark of A. S. Wightman [Lille Conference Report, (1957)] may be mentioned. If the commutation relation $[\phi(x), \phi(y)]=0$, $(x-y)^2<0$ is weakened to $[\phi(x), \phi(y)]=0$, $(x-y)^2<-l_0^2$, the analyticity properties of vacuum to vacuum matrix elements in local field theories ensure that the second relation implies the first. *A. Salam* (London).

Watanabe, Satosi. Chirality of K particle. Phys. Rev. (2) **106** (1957), 1306–1315.

In the two-state theory of the neutrino, this particle is characterised by one of the eigenvalues of the "chirality" operator γ_5. This operator anticommutes with the parity operator. The concept of "chirality" is generalized so that it can apply to bosons. Essentially, a scalar field with definite "chirality" turns out to be a special superposition of conventional scalar and pseudo-scalar quantities. In this respect the author's work has similarities to the ideas of Gell-Mann and Pais [Phys. Rev. (2) **97** (1955), 1387–1389]; MR **17**, 923]. It is suggested that the "chirality" operator may have a close connection with "strangeness". *A. Salam* (London).

Zaikov, Rashko. Quantum mechanical characteristics and elementary particles. C. R. Acad. Bulgare Sci. **10** (1957), 101–103. (1 plate). (Russian summary)

The scheme for classification of elementary particles proposed by Salam and Polkinghorne [Nuovo Cimento (10) **2** (1955), 685–690] is extended to include leptons. The following empirical mass formulae are shown to hold

$$M\sim 3[8r_1+r_2]m_e \text{ for bosons;}$$

$$M\sim 3[8(r_1+\tfrac{1}{2})+r_2]m_e \text{ for fermions.}$$

Here r_1 is a specific integer (11 for pions, 76 for nucleons, 90 for Λ^0 etc.) and r_2 takes the values 0, 1, 2, 3.

A. Salam (London).

d'Espagnat, Bernard. Sur les interactions faibles baryons-mésons π. C. R. Acad. Sci. Paris **245** (1957), 894–896.

If the Λ, Σ mass difference is neglected, the combination $\Lambda+\tau\cdot\Sigma$ may be considered as forming a second rank spinor in isotopic space. Provided that certain coupling constants are equal, the $\Lambda N\pi$ and $\Sigma N\pi$ decay interactions can be combined and written in terms of this spinor. This gives an alternative formulation of the $|\Delta I|=\frac{1}{2}$ rule for weak decays [see R. Gatto, Nuovo Cimento (10) **3** (1956), 318–335; G. Wentzel, Phys. Rev. (2) **101** (1956), 1215–1216; G. Takeda, ibid. **101** (1956), 1547–1551]. *A. Salam* (London).

McLennan, James A., Jr. Parity nonconservation and the theory of the neutrino. Phys. Rev. (2) **106** (1957), 821–822.

The equivalence of the free field equations for two component and Majorama theory is exhibited explicitly. A transformation is defined which involves complex conjugation and may be interpreted as space reflection. The free two component theory is invariant under this transformation. *P. T. Matthews* (London).

Marx, G.; and Györgyi, G. Symmetry operations of the Hilbert space and selection rules concerning the interactions of baryons and mesons. Nuovo Cimento (10) **5** (1957), supplemento, 159–181.

Selection rules and relations between cross-sections, implied by change-independence and charge-conjugation invariance, are derived for the following processes: Anti-proton and anti-hyperon annihilation with production of pions or K-mesons; hyperon, K-meson and anti-hyperon production. Some sections of this paper are out of date because they assume that K-mesons are parity doublets, but the necessary changes are easy to make. *J. C. Taylor* (London).

Tzou, K. H. Représentation corpusculaire du champ vectoriel et comparaison au champ de spin maximum 1 de la théorie de fusion. J. Phys. Radium (8) **18** (1957), 619–624.

The theory of a vector field (comprising particles of spin 0 and spin 1) is written in such a form as to be directly comparable with de Broglie's theory of a particle of maximum spin 1. It is concluded that the internal spin structures of the theories differ, although they can be decomposed into the same spin states. *H. W. Lewis.*

Karpman, V. I. On the S-matrix for particles with arbitrary spin. Soviet Physics. JETP **3** (1957), 934–940.

This is a discussion of the perturbation theory for particles with arbitrary spin. The properties of the singular functions for such particles are discussed. It is shown that the elements of the S-matrix may be found by the Feynman rules. *Author's summary.*

Zaretskii, D. F.; and Shut'ko, A. V. Quasi-magnetic interaction of the spin of a nucleon with the rotation of the nucleus. Soviet Physics. JETP **5** (1957), 323–325.

Fradkin, E. E. Particle with spin 3/2 in an electromagnetic field. Soviet Physics. JETP **5** (1957), 298–299.

See also: Optics, Electromagnetic Theory, Circuits: Ichikawa. **Relativity:** Wigner.

Relativity

Wigner, Eugene P. Relativistic invariance and quantum phenomena. Rev. Mod. Phys. **29** (1957), 255–268.

In the first part of this paper the author points out that the question of consistency between the special theory of relativity and quantum mechanics can be formulated and that the two theories use common concepts such as measurements of position and momenta. The second part of the paper is devoted to showing that the relationship between quantum theory and general relativity is not as clear logically.

The first part of this paper contains a discussion of the fact that particles with zero rest mass have only two directions of polarization no matter what their spin is, whereas particles with non-zero rest mass and spin S have $2S+1$ states of polarization. This fact is related to the fact that the statement that the spin is parallel to the velocity is a relativistically invariant statement only for particles with zero rest mass. This part also contains a discussion of reflection symmetry.

The second part of the paper is devoted to a discussion of the quantum limitations of the concepts of general relativity. It considers quantum limitations on the accuracy of the conversion of timelike measurements into space-like measurements, the limitations of accuracy of a clock, and the measurement of curvature of a region of space-time. The results are based on calculations involving one space dimension and time, and may have to be modified when three spatial dimensions are used. Most of the conclusions concerning the general theory of relativity were arrived at in collaboration with Dr. H. Salecker, and a detailed report of these in another publication is promised. *A. H. Taub* (Urbana, Ill.).

Gürsey, Feza. On some conformal invariant world-lines. Rev. Fac. Sci. Univ. Istanbul. Sér. A. **21** (1956), 129–143 (1957). (Turkish summary)

Referring to papers by J. Haantjes [Nederl. Akad. Wetensch., Proc. **44** (1941), 814–824; MR **3**, 189] and E. L. Hill [Phys. Rev. (2) **72** (1947), 143–149; MR **9**, 107], the author derives the relations between the conformal curvatures h_1, h_2 of a world line and its Frenet curvatures, and in particular considers world lines for which $h_1=h_2=0$ in connection with the equations of motion of a radiating electron. *J. L. Synge* (Dublin).

Mariot, Louis. Champ électromagnétique singulier complètement intégrable. C. R. Acad. Sci. Paris **245** (1957), 1386–1388.

This paper is one of a sequence dealing with singular electromagnetic fields, i.e. fields satisfying $F_{\alpha\beta}F^{\alpha\beta}=0$, $F_{\alpha\beta}{}^*F^{\alpha\beta}=0$ [same C. R. **238** (1954), 2055–2056; **239** (1954), 1189–1190; **241** (1955), 175–176; **245** (1957), 630–631; MR **15**, 995; **16**, 756; **17**, 110]. In Riemannian space-time, let λ^α be a field of null vectors such that $\lambda_\alpha dx^\alpha=d\theta$ is an exact differential, and let A^α be a vector field orthogonal to λ^α. Then $F_{\alpha\beta}=(\lambda_\alpha A_\beta-\lambda_\beta A_\alpha)\cos\theta$ is a singular electromagnetic field. The author shows that this field satisfies Maxwell's equations in vacuo ($\nabla_\alpha F^{\alpha\beta}=0$, $\nabla_\alpha{}^*F^{\alpha\beta}=0$), but the details of the argument are not clear to the reviewer. *J. L. Synge* (Dublin).

Pham, Mau Quan. Inductions électromagnétiques en relativité générale et principe de Fermat. Arch. Rational Mech. Anal. **1** (1957), 54–80.

In a charged conducting fluid as treated in previous papers [J. Rational Mech. Anal. **5** (1956), 473–538; C. R. Acad. Sci. Paris **242** (1956), 465–467; MR **17**, 1144], the two electromagnetic tensors and the 4-velocity are connected by the constitutive equations $G_{\alpha\beta}u^\alpha=\varepsilon H_{\alpha\beta}u^\alpha$, $\mu\mathring{G}_{\alpha\beta}u^\alpha=\mathring{H}_{\alpha\beta}u^\alpha$, where ε=dielectric coefficient, μ=magnetic permeability; these may be solved, yielding

$$\mu G^{\alpha\beta}=H^{\alpha\beta}+(1-n^2)(H^{\sigma\alpha}u_\sigma u^\beta-H^{\sigma\beta}u_\sigma u^\alpha),$$

where $n^2=\varepsilon\mu$. The Cauchy problem is examined, and the characteristic 3-spaces $f(x)=0$ are found to satisfy $\bar{g}^{\alpha\beta}\partial_\alpha f\partial_\beta f=0$, $\bar{g}^{\alpha\beta}=g^{\alpha\beta}-(1-n^2)u^\alpha u^\beta$. This suggests the use of the associated metric $\bar{g}_{\alpha\beta}=g_{\alpha\beta}-(1-n^{-2})u_\alpha u_\beta$, in terms of which Maxwell's equations admit a simple form, while the bicharacteristics (electromagnetic rays) are the null geodesics with respect to $\bar{g}_{\alpha\beta}$ [cf. W. Gordon, Ann. Physik (4) **72** (1923), 421–456; N. L. Balazs, J. Opt. Soc. Amer. **45** (1955), 63–64; MR **16**, 872]. Steady motions of a perfect charged fluid are studied, and a generalised form of Fermat's principle is obtained for the spatial projections of the electromagnetic rays.

J. L. Synge (Dublin).

Takeno, Hyôitirô; and Ueno, Yoshio. On the wave theory of light in general relativity. III. Electromagnetic four potential. Progr. Theoret. Phys. **15** (1956), 322–332.

[For Parts I and II, written by Y. Ueno, see same journal **10** (1953), 442–450; **12** (1954), 461–480; MR **15**, 655; **16**, 872.] For an electromagnetic field satisfying the usual Maxwell equations in curved space-time (with given metric, independent of the field), the authors discuss the 4-potential in relation to various normalising conditions and gauge transformations. The case of statical space-time with spherical symmetry is given particular attention. They conclude that the introduction of a 4-potential scarcely brings advantages in curved space-time, and it may be better to do without it, as in the work of L. Infeld and J. Plebanski [Acta Phys. Polon. **12** (1953), 123–134; Proc. Roy. Soc. London. Ser. A **222** (1954), 224–227; MR **15**, 489, 765]. *J. L. Synge.*

Lanczos, Cornelius. Electricity and general relativity. Rev. Mod. Phys. **29** (1957), 337–350.

In previous papers [Phys. Rev. (2) **39** (1932), 716–736; **61** (1942), 713–720; Rev. Mod. Phys. **21** (1949), 497–502; MR **4**, 56; **11**, 548] the author dealt with action principles in which the integrand is quadratic in the curvature components. In the present paper he pursues this theme, first presenting in general form a process of Hamiltonization for a wide class of Lagrangians, then making a special choice of Lagrangian, modifying this Lagrangian so as to include electricity, and finally seeking a physical interpretation of the theory, which is compared with the theories of H. Weyl [Math. Z. **2** (1918), 384–411; Ann. Physik (4) **59** (1919), 101–133; Phys. Z. **22** (1921), 473–480] and W. Pauli [ibid. **20** (1919), 457–467; Encykl. Math. Wiss., Bd. V, T. 2, Teubner, Leipzig, 1921, p. 759]. The theory is set in Riemannian space-time, with metric tensor g_{ik} and Ricci tensor R_{ik}. Starting with a Lagrangian $L_0=f(R_{ik}, g^{ik})$, one obtains a variational equation equivalent to $\delta\int L_0 d\tau=0$ if one substitutes a new Lagrangian $L=p^{ik}(R_{ik}-w_{ik})-f(w_{ik}, g^{ik})$, and gives independent variations to the 30 quantities p^{ik}, w_{ik}, g^{ik} (symmetric in i, k). Further, one gets an equivalent equation on changing L to

$$L=\tfrac{1}{2}p^{ik}(\partial_k\Gamma_{ia}{}^\alpha+\partial_i\Gamma_{ka}{}^\alpha-2\partial_\alpha\Gamma_{ik}{}^\alpha)+\gamma_{ik}{}^m\partial_m g^{ik}-H_1-H_0,$$

where

$$H_1=p^{ik}(\Gamma_{ik}{}^\beta\Gamma_{\beta\alpha}{}^\alpha-\Gamma_{i\alpha}{}^\beta\Gamma_{k\beta}{}^\alpha)-\gamma_{ik}{}^m(\Gamma_{m\alpha}{}^i g^{k\alpha}+\Gamma_{m\alpha}{}^k g^{i\alpha}),$$

$$H_0=p^{ik}w_{ik}-f(w_{ik}, g^{ik}),$$

and giving independent variations to the 110 quantities p^{ik}, w_{ik}, g^{ik}, $\Gamma_{ik}{}^m$, $\gamma_{ik}{}^m$ (all suffixes run from 1 to 4). However, the variation of w_{ik} gives $p^{ik}=\partial f/\partial w_{ik}$, and if these equations are solved for w_{ik} and the solution substituted, H_0 may be written as a function of (p^{ik}, g^{ik}), and there are only 100 field variables left (elimination of w_{ik}). The resultant partial differential equations are

$$\partial_m g^{ik}+\Gamma_{m\alpha}{}^i g^{k\alpha}+\Gamma_{m\alpha}{}^k g^{i\alpha}=0,$$

$$-\gamma_{ik,\alpha}{}^\alpha=\partial H_0/\partial g^{ik}-\tfrac{1}{2}H_0 g_{ik},$$

$$\tfrac{1}{2}(\partial_k\Gamma_{i\alpha}{}^\alpha+\partial_i\Gamma_{k\alpha}{}^\alpha-2\partial_\alpha\Gamma_{ik}{}^\alpha)+\Gamma_{i\alpha}{}^\beta\Gamma_{k\beta}{}^\alpha-\Gamma_{ik}{}^\beta\Gamma_{\beta\alpha}{}^\alpha=\partial H_0/\partial p^{ik}.$$

They are linear in the derivatives, and have constant coefficients (the comma indicates covariant derivative).

As a quadratic Lagrangian, the author uses $L_0=\frac{1}{2}(R_{ik}R^{ik}+\beta R^2)$, where β is some unspecified numerical constant, and obtains

$$H_0=\tfrac{1}{2}(p^{ik}p_{ik}+\sigma p^2),\quad \sigma=-\beta(1+4\beta)^{-1}.$$

Application of the method described above in a simplified form (without the Γ's or γ's), in fact, with $L=p^{ik}R_{ik}-H_0$, gives the following 20 equations for g_{ik}, p_{ik}:

$$R_{ik}=p_{ik}+\sigma p g_{ik},$$

$$D^{ik}(p)-(p^{i\alpha}p_\alpha{}^k-\tfrac{1}{4}p^{\alpha\beta}p_{\alpha\beta}g_{ik})-\sigma p(p^{ik}-\tfrac{1}{4}pg^{ik})=0,$$

where $D_{ik}(p)=\frac{1}{2}(g^{\alpha\beta}p_{ik,\alpha\beta}-p^\alpha{}_{i,k\alpha}-p^\alpha{}_{k,i\alpha}+p^{\alpha\beta}{}_{,\alpha\beta}g_{ik})$. The cosmological equations $R_{ik}=(1+4\sigma)\lambda g_{ik}$, with λ an arbitrary constant, represent an exact solution of the field equations.

To bring in electricity, a vector φ_i, and a Lagrangian of the form $L=p^{ik}R_{ik}^*-(H_0+H_1)$ are introduced, where

$$R_{ik}^*=R_{ik}-\tfrac{1}{2}(\varphi_{i,k}+\varphi_{k,i}+\varphi_i\varphi_k)+\tfrac{1}{2}[(1+2\sigma)\varphi_{,\alpha}{}^{\alpha}-\sigma\varphi^{\alpha}\varphi_{\alpha}]g_{ik},$$

$$H_0=\tfrac{1}{2}(p^{ik}p_{ik}+\sigma p^2),$$

$$H_1=\tfrac{1}{8}F_{ik}F^{ik}+\tfrac{1}{2}(1+\sigma)(\varphi_{,\alpha}{}^{\alpha}-\tfrac{1}{2}\varphi^{\alpha}\varphi_{\alpha})^2,$$

$$F_{ik}=\partial_k\varphi_i-\partial_i\varphi_k.$$

The role of φ_i is discussed, and it is identified, not with 4-potential, but with electric current. The paper ends with a physical interpretation, in which λ is considered to be very large, and the universe is regarded as statistically smooth, but microscopically a seething cauldron of ultra-high-frequency oscillations (tremorfield). *J. L. Synge.*

Rosen, Nathan; and Shamir, Hadassah. Gravitational field of an axially symmetric system in first approximation. Rev. Mod. Phys. **29** (1957), 429–431.

A simple physical system with axial symmetry, referred to spherical polar coordinates, is studied, special attention being paid to the possibility of radiation of gravitational waves. A solution of the field equations for empty space is sought, where the coefficients of the line-element are assumed to differ from those of "flat" space-time by quantities of the first order of smallness only. The energy-momentum-stress tensor is evaluated on the basis of a simple oscillatory model of the source. It is found that, in the linear approximation, the resulting field equations admit a solution describing gravitational waves emitted by an oscillating physical system. The rate of emission of energy is also evaluated. The authors' result appears to be in agreement with earlier deductions [e.g. Landau and Lifshitz, The classical theory of fields, OGIZ, Moscow-Leningrad, 1948, p. 331; MR **11**, 410; **13**, 289].

H. Rund (Durban).

Schmutzer, Ernst. Beitrag zur projektiven Relativitätstheorie. Z. Physik **149** (1957), 329–339.

The author endeavours to develop a generalization of the projective theory of relativity by abandoning the usual holonomicity assumptions with respect to the coordinate frames of the 5-dimensional manifold. {The geometrical significance of this procedure is not made clear, nor are the underlying assumptions regarding the general nature of these manifolds clearly specified}. A generalised covariant derivative is defined; this is similar to the differentiation process commonly used in projective theories. The resulting deviation of the present theory from these earlier theories is studied in detail, giving rise to a new analysis of the electromagnetic field.

The general physical interpretation differs from that of Jordan, although a significant feature of the author's approach is the fact that the invariant J is regarded as variable [as in the work of Jordan; see G. Ludwig, Fortschritte der projektiven Relativitätstheorie, Vieweg, Braunschweig, 1951; MR **14**, 213]. The paper concludes with some cosmological considerations. *H. Rund.*

Moffat, John. The static spherically symmetric solutions in a unified field theory. Proc. Cambridge Philos. Soc. **53** (1957), 489–493.

The author finds the solution of the title for the unified field equations proposed by him in two previous papers [same Proc. **52** (1956), 623–625, MR **18**, 332; ibid. **53** (1957), 473–488]. The solution is the Schwarzschild solution, but with a complex constant of integration $m+ie$ in place of the usual real constant m. He identifies m with the mass of the central singularity, e with its charge. {It is not clear to the reviewer that the latter identification is justified by the author's previous work.}

F. A. E. Pirani (London).

See also: Statistical Thermodynamics and Mechanics: Pathria. **Quantum Mechanics:** Blohincev.

Astronomy

Schmeidler, F. Über eine Funktionalgleichung der Meridianastronomie. Astr. Nachr. **283** (1957), 241–244.

Radsievsky, V. V.; and Gelfgat, B. E. The restricted problem of two bodies of variable mass. Astr. Ž. **34** (1957), 581–587. (Russian. English summary)

Several authors have recognized the importance of the mass-variation in the cosmogonical problems. The present authors investigate the case in which the mass of the sun varies according the law: $dm/dt=-\alpha m^n$. A transformation of the coordinates and of the time reduces the problem of the motion of the planet to the motion under the influence of the gravitational force and of the quasi-elastic and friction forces. The solution of the problem for any n can be found, if the problem is solved for $1\leqq n\leqq 3$. The equation of the approximate orbit has the form of the equation of an ellipse. Such an "ellipse" was originally introduced by Duboshin. The authors found a new, strictly integrable, case of motion in a gravitational and resisting atmosphere. If the mass of the sun is varying according to the law $m=m_0e^{-\alpha t}$ and the density of the medium is constant, then a periodic motion is possible and the orbit is a conic section. *P. Musen.*

Prendergast, Kevin H. The equilibrium of a self-gravitating incompressible fluid sphere with a magnetic field. I. Astrophys. J. **123** (1956), 498–507.

The work of Ferraro [Astrophys. J. **119** (1954), 407–412; MR **16**, 183] and Roberts [ibid. **122** (1955), 508–512; MR **17**, 906] is extended by the introduction of a toroidal as well as a poloidal field, whence it is shown that the equations of hydromagnetic equilibrium can be satisfied for a spherical configuration without having to postulate force-free fields. The strength of the magnetic field is limited by the Chandrasekhar-Fermi condition for dynamic stability. The questions of the uniqueness and the stability of the equilibrium configuration are left for future investigation. *K. C. Westfold.*

Bhatnagar, P. L.; and Nagpaul, S. R. Radial pulsations of an infinite cylinder with finite conductivity immersed in magnetic field. Z. Astrophys. **43** (1957), 273–288.

The radial pulsations of an infinite homogeneous cylinder, surrounded by an uniform magnetic field which is parallel to the axis of the cylinder, are investigated.

In part I the usual assumption of infinite electrical conductivity in the cylinder is made, and inside the cylinder the magnetic field considered is given by

$$H^2=H_s^2+(H_0^2-H_s^2)(1-x^2),$$

where H_s and H_0 are the magnetic field at the surface and at the axis of the cylinder respectively, and x is the distance from the axis, where the radius of the cylinder has length unity. From the equations given by Chandra-

sekhar and Fermi [Astrophys. J. **118** (1953), 116–141; MR **15**, 168], the differential equation for small oscillations and the corresponding formula for the variation of the magnetic field are deduced. A series expansion for the displacement function is derived, and the boundary condition of no variation of the total pressure at the surface of the cylinder gives a series from which the characteristic values (essentially the characteristic frequencies) are obtained. For some selected cases numerical values are given for the first three characteristic values as well as the displacement function and the corresponding relative variation of the magnetic field. It is emphasized that the displacement functions in general are not polynomials if the magnetic field at the surface of the cylinder is non-vanishing. Some exceptions from this rule are treated at some length.

In part II the condition of infinite electrical conductivity is dropped. The magnetic field is considered uniform both outside and inside the cylinder. The equations for the oscillations are deduced in about the same way as in part I. In the expressions for the characteristic values, the displacement functions and the variation of the magnetic field, terms of second and higher order with respect to the reciprocal of the electrical conductivity are omitted. Then it is found that the oscillation periods are unchanged, but the finite electrical conductivity causes a damping of the oscillation and, furthermore, a variation of phase, which is different for the mechanical and magnetic pulsations. *E. Lyttkens* (Uppsala).

Erickson, William C. A mechanism of non-thermal radio-noise origin. Astrophys. J. **126** (1957), 480–492.

The discovery of strong sources of radio noise in localized regions of interstellar space has opened a new field of astronomical research, known as radio-astronomy. It is undoubtedly the case that more than one mechanism exists for the generation of electromagnetic radiation of this type. The author has examined a mechanism which appears to be appropriate in the case of sources consisting of huge gas clouds with large internal velocity distributions, which have been observed to occur. It is proposed that small grains of solid material are bombarded with high speed protons of the gas cloud, and so gain translational and rotational energy. When the angular velocities of the rotational motions rise to values characteristic of the observed radio frequencies, the grains which have electric or magnetic moments can contribute to the radio noise by emission of electromagnetic fields. The major interest in the paper at present lies in its estimates of the physical orders of magnitude associated with the mechanism. From the mathematical point of view it may lead to a novel type of statistical problem associated with the Boltzmann diffusion equation. *E. L. Hill.*

See also: Mechanics of Particles and Systems: Sedov; Mihailović. **Fluid Mechanics, Acoustics:** Rogers.

Geophysics

Watson, G. S.; and Irving, E. Statistical methods in rock magnetism. Monthly Not. Roy. Astr. Soc. Geophys. Suppl. **7** (1957), 289–300.

This paper describes the statistical techniques available to the experimenter in palaeomagnetic work. The theory of these methods is based on an assumed probability distribution of errors. It is shown that the mathematical requirements of this distribution are obeyed by the observations from rock samples which are known to possess a stable magnetization; observations on rocks with unstable magnetization however do not conform to it. A theoretical derivation is given for this probability distribution.

The problem of estimating the mean direction of magnetization of a geological formation has in recent years become a matter of the greatest geophysical interest since it is from such estimates that the position of the pole of the Earth in past geological ages is determined. This problem is largely one of the judicious choice of samples and a procedure is suggested whereby such estimates may be achieved with the greatest sample economy.

Authors' summary.

Kurbatkin, G. P. Determination by the methods of hydrodynamics of the annual course of the temperature of air on sea level. Izv. Akad. Nauk SSSR. Ser. Geofiz. **1957**, 228–243. (Russian)

En utilisant les recherches de E. N. Blinova [Izv. Akad. Nauk SSSR. Ser. Geograf. Geofiz. **1** (1947)], l'auteur expose une théorie plus détaillée qui va permettre d'étudier l'effet de condensation et les changements brusques de la courbure des isothermes au voisinage des contours. Ceci permet de calculer les propriétés des courants thermiques au voisinage du sol en fonction des coordonnées géographiques. L'auteur compare ses résultats théoriques avec les données d'observation. *M. Kiveliovitch.*

Duhin, S. S.; and Deryagin, B. V. Theory of interaction of evaporating or growing drops at large distances. Dokl. Akad. Nauk SSSR (N.S.) **112** (1957), 407–410. (Russian)

L'étude isothermique du mouvement d'une goutte dans un champ de diffusion a permis de se rendre compte des forces d'attraction et de répulsion qui agissent entre les gouttes du brouillard ou entre une goutte et une surface humide. On a trouvé qu'en première approximation les forces de diffusion sont compensées par le courant de Stefan de façon que la vitesse de mouvement de la goutte par rapport à la surface de passage de phase est voisine de zéro. L'auteur montre que parallèlement avec la diffusion agit un autre processus de conduction de chaleur qui détruit la compensation obtenue en première approximation; ceci permet aux forces de diffusion d'influencer la condensation et la coagulation de l'aérosol.

M. Kiveliovitch (Paris).

Kohlsche, Kurt. Bemerkungen zur Verwendung des Begriffes der Informationsentropie für Vorhersageprobleme und Ableitung einer Beziehung zur Bewertung von Prediktoren. Z. Meteorol. **11** (1957), 193–199.

Nach einer kritischen Betrachtung über die Bedeutung des Begriffes der Informationsentropie für die meteorologische Vorhersage wird ein Vorhersageverfahren entwickelt, das bei vorgegebenen Prediktoren eine maximale Trefferzahl ergibt. Es wird direkt der prozentuale Treffergewinn gegenüber einer rein klimatologischen Vorhersage bestimmt. Da dieser aber wesentlich von der Klasseneinteilung des vorherzusagenden Merkmals (Genauigkeit der Vorhersage) abhängt, wird noch ein von der Klasseneinteilung weitestgehend unabhängiges Bewertungsverfahren entwickelt. *Zusammenfassung des Autors.*

Grant, Fraser. A problem in the analysis of geophysical data. Geophysics **22** (1957), 309–344.

Observed data mapped in two dimensions are ex-

pressed as a linear combination of orthogonal polynomials (the "trend") and "residuals", for which the coefficients are determined by the method of least squares. The method is discussed in the light of stochastic theories, and an example is given of the regional reduction of a gravity survey. *A. Marussi* (Trieste).

Baarda, W. Some remarks on the computation and adjustment of large systems of geodetic triangulation. Bull. Géodésique **1957**, 20–49.

The adjustment of large continental nets of triangulation is divided into two phases, the first of which concerns mainly the geometric conditions arising from the individual networks, the second the conditions arising from their junction and from the differences between geoid and spheroid, as computed from gravimetric observations. *A. Marussi* (Trieste).

★**Fiala, František. Matematická kartografie. [Mathematical cartography.]** Nakladatelstvi Československé Akademie Věd, Prague, 1955. 288 pp. (4 tables). 22.00 Kčs.

★**Drake, Joh. Taschenbuch für Vermessungsingenieure. Zweite, erweiterte und berichtigte Auflage.** VEB Verlag Technik Berlin, 1955. 200 pp.

See also: Differential Geometry: Šmahel.

OTHER APPLICATIONS

Economics, Management Science

Georgescu-Roegen, Nicholas. Threshold in choice and the theory of demand. Econometrica **26** (1958), 157–168.

The author generalizes the usual assumptions about binary choice among commodity bundles by introducing probabilistic choices. Let $\omega(A, B)$ be the probability that A is chosen over B. The following assumptions are made: (1) If A is greater than B in a vector sense, then $\omega(A, B)=1$; (2) $\omega(X, A)$ is a continuous function of X for $X\neq A$; (3) if A is less than B in a vector sense, then $\omega(A, C)\leqq\omega(B, C)$, where the equality implies either $\omega(A, C)=1$ or $\omega(B, C)=0$; (4) if $\omega(A, B)=\omega(B, C)=p\geqq\frac{1}{2}$, then $\omega(A, C)\geqq p$; (5) if C is a convex combination of A and B, then $\omega(A, B)\leqq\omega(C, B)$. From these axioms, a number of properties of $\omega(A, B)$ are deduced, of which the following is typical: if $\omega(A, B)\geqq\frac{1}{2}$ and $\omega(B, C)\geqq\frac{1}{2}$, then $\omega(A, C)\geqq\min[\omega(A, B), \omega(B, C)]$, with the equality sign implying that $\omega(A, B)=\omega(B, C)$.

These axioms do not imply anything about choices made from sets containing more than two elements; such choices are, of course, essential for a theory of consumer behavior. Two additional axioms are suggested for this purpose: (1) for any $A_1, \cdots, A_n$, $\omega(A_2, \cdots, A_n)=\omega(A_1, A_2, \cdots, A_n)+\omega(A_2, A_1, A_3, \cdots, A_n)+\omega(A_2, A_3, A_1, \cdots, A_n)+\cdots+\omega(A_2, A_3, \cdots, A_1, A_n)+\omega(A_2, A_3, \cdots, A_n, A_1)$, where $\omega(A_1, \cdots, A_n)$ is the probability that the preference ordering is $A_1, \cdots, A_n$; (2) $\omega(A_1|A_1, \cdots, A_n)=\sum\omega(A_1, A_{i_2}, \cdots, A_{i_n})$, where $\omega(A_1|A_1, \cdots, A_n)$ is the probability of choosing A_1 from the set $[A_1, \cdots, A_n]$ and the sum extends over all permutations $(i_2, \cdots, i_n)$ of $(2, \cdots, n)$. Then it is shown that the probability of choice $\omega(A_1|A_1, \cdots, A_n)$ is not capable of expression as a function of the probabilities $\omega(A_i, A_j)$. *K. J. Arrow.*

Wold, H. O. A.; and Whittle, P. A model explaining the Pareto distribution of wealth. Econometrica **25** (1957), 591–595.

Let $f(x, t)$ be the frequency distribution of estates of size x at time t. It is supposed that estates above a lower limit K will grow at a proportional rate $\beta\Delta+o(\Delta)$ in time interval Δ and that any holder has a chance $\gamma\Delta+o(\Delta)$ of dying in this interval, in which event his estate is divided equally among n heirs. In the interval $x\geqq K$, the density then satisfies the equation

$$(1)\qquad \partial f/\partial t=-(\beta+\gamma)f-\beta x(\partial f/\partial x)+\gamma n^2 f(nx, t).$$

The number of individuals with estates above K will grow at the rate $\gamma(n-1)$. Let $R(t)$ be the number whose estates are below K; then it is assumed that this number grows naturally at the same rate, but a fraction λ pass into the upper group, and that there are also added to the lower group those heirs of upper-class estates of size less than K. Then

$$(2)\qquad \frac{\partial R}{\partial t}=[\gamma(n-1)-\lambda]R(t)+n\gamma\int_K^{nK} f(y, t)dy.$$

There is also a consistency condition between the equations (1) and (2). Among the solutions to these equations is one in which $f(x, t)=Ax^{-\alpha-1}e^{\theta t}$, that is, the large estates have a Pareto distribution. The characteristic equation becomes,

$$(3)\qquad (\beta/\gamma)\alpha=n(1-n^{-\alpha}).$$

When (3) is satisfied, $\theta=\gamma(n-1)$, so that the distribution is stationary. It is argued that empirically plausible values for the parameters α, β, γ, and n are consistent with (3). *K. J. Arrow* (Stanford, Calif.).

Uzawa, Hirofumi. On intertemporal efficiency conditions of capital accumulation. I. Ann. Inst. Statist. Math., Tokyo **7** (1956), 195–204.

This paper summarizes and restates in set-theoretic terms some results found by P. A. Samuelson and the reviewer (in an unpublished paper to appear in a forthcoming book with R. Dorfman on Linear Programming and Economic Analysis) using direct variational methods. Let $x(t)$ represent an n-dimensional vector of nonnegative commodity stocks, $T(x(t))$ represent the set of stock vectors $x(t+1)$ obtainable from $x(t)$, and $E(x(t))$ represent the efficient set in $T(x(t))$. A path $x(t)$, $x(t+1)$, $\cdots$, $x(t+h)$ is feasible if $x(t+k)\in T(x(t+k-1))$, $k=1, \cdots, h$, and efficient if no other feasible path of length h leads to a terminal vector componentwise larger than $x(t+h)$. Say then that $x(t+h)\in E^h(x(t))$. Under certain regularity and convexity assumptions on the sets T, the main results quoted by the author are as follows. (A) If $x(t+1)\in E(x(t))$, then there exists an $x(t+2)$ such that $x(t)$, $x(t+1)$, $x(t+2)$ is efficient. (B) A path of length 2 is efficient if and only if the marginal rate of substitution between any two stocks in $x(t+1)$ regarded as outputs of the previous period equals their marginal rate of substitution as inputs for the next period. (C) A path of length h is efficient if and only if each of the 2-period paths of which it is composed is efficient. (D) There exists at least one efficient path of length h such that $x(t+k)=\lambda x(t+k-1)$, $k=1, \cdots, h$, with λ a positive constant. Of these, the author proves (A),

the equivalent of (B), and the necessity half of (C), using positive price vectors and their associated separation properties. *R. Solow* (Cambridge, Mass.).

Frisch, Ragnar. Sur un problème d'économie pure. Metroecon. **9** (1957), 79–111.

This is a re-publication from Norsk Mat. Forenings Skr. Ser. I. no. **16** (1926). It deals with the axiomatic foundations of utility analysis in economics, and with the problem of measurability. Using the simplifying notion of "independent commodities", the author derives an index for the marginal utility of money. The use of the theory is illustrated by application to French statistical data for 1920–22. *T. Haavelmo* (Chicago, Ill.).

★ **Quinkert, Werner. Die kollektive Risikotheorie unter Berücksichtigung schwankender Grundwahrscheinlichkeiten mit endlichem Schwankungsbereich.** Inaugural-Dissertation zur Erlangung des Doktorgrades der Mathematisch-Naturwissenschaftlichen Fakultät der Universität zu Köln, 1957. iv+40 pp.

H. Ammeter hat in der Risikotheorie schwankende Grundwahrscheinlichkeiten zur Beschreibung der Frequenzfunktion der Schadenfälle eingeführt und als Primärverteilung dabei eine Γ-Verteilung benutzt. Der Verfasser führt analoge Untersuchungen mit Hilfe einer B-Verteilung erster Art als Primärverteilung durch und vergleicht sie mit den Ergebnissen von Ammeter.
W. Saxer (Zürich).

Fulkerson, D. R.; and Johnson, S. M. A tactical air game. Operations Res. **5** (1957), 704–712.

A discrete, linear model of a tactical air war is formulated as a multi-move game. The symmetric case in which the attrition rates are the same for both sides is solved for both finite and infinite campaigns. (Author's summary.)
T. L. Saaty (Washington, D.C.).

★ **Morse, Philip M. Queues, inventories and maintenance. The analysis of operational systems with variable demand and supply.** Publications in Operations Research, Operations Research Society of America, No. 1. John Wiley & Sons, Inc., New York; Chapman & Hall, Ltd., London, 1958. ix+202 pp. $6.50.

Contents: (1) Representation in terms of probabilities (state probabilites); (2) probabilistic description of arrival and service time (service time distribution, arrival distribution; (3) single exponential channel (equations of detailed balance, balance between service cost and customers lost, infinite queues, balance between mean wait and service cost, effect of customers' impatience; (4). multiple exponential channels (service channels in parallel, optimization of the number of channels, sequential service lines); (5) simulation of non-exponential distributions (simulation by series arrangements, effect of service-time distributions, effect of arrival-time distributions, simulation by parallel arrangement, examples with hyper-exponential distributions); (6) general considerations, transient solutions (the general equations of detailed balance, an example of transient behavior, transient solutions for finite queues); (7) single channel, infinite queues (the general equations, the single exponential channel, effect of service-time distribution, details of $k=2$ case, hyper-exponential service, non-Poisson arrivals, details of the $l=2$ case, both service and arrivals non-exponential, hyper-Poisson arrivals, example of service line); (8) multiple channels, infinite queues (exponential channels, effect of arrival-time distribution, effects of service time distributions); (9) queue discipline and priorities (single channel random access to service, single channel with priorities, the effect of priorities on average delay, multiple channels, single versus multiple queues); (10) problems of inventory control (the effects of variance of demand and supply, re-order for each item sold, determination of optimum inventory, re-order when out of stock, advisability of buffer stock, re-order by batches); (11) maintenance of equipment (breakdown-time distribution, single machine, optimum repair effort, preventive maintenance, when preventive maintenance is advisable, many machines-single repair crew, many machines-more than one repair crew).

The analysis is based upon the following distribution functions: Erlang distribution function, $e_n(x)=x^n\varepsilon^{-x}/n!$; $E_m(x)=\sum_{n=0}^{m} e_n(x)$; $D_m(x)=\sum_{n=0}^{m} E_n(x)/(m+1)$. Hyper-exponential distribution, $2\sigma^2\varepsilon^{-2\sigma x}+2(1-\sigma)^2\varepsilon^{-2(1-\sigma)x}$.

Tables are provided for: $ke_{k-1}(kx)$: $x=0,\ (0.1)2.0$; $k=1,\ 2,\ 3,\ 4(2)12,\ 16,\ 20$. $e_m(mx)$, $E_m(mx)$, $D_m(mx)$: $x=0.1(0.1)2.0$; $m=1(1)6(2)16,\ 20$. Also for the hyper-exponential distribution: for $j=(1-2\sigma+2\sigma^2)/2\sigma(1-\sigma)=1,\ 2,\ 4,\ 10,\ 20$; $x=0(0.2)2.0$; and for $Q_m(mx)=e_m(mx)/D_{m-1}(mx)$ for $x=0(0.1)2.0$; $m=1(1)4(2)12,\ 16,\ 20$.
G. Tintner (Ames, Iowa).

Dressin, S. A.; and Reich, E. Priority assignment on a waiting line. Quart. Appl. Math. **15** (1957), 208–211.

There are r classes of customers with priorities $p(=1,\ 2\cdots r)$. Customers of priority p arrive as a Poisson process with an interval period distribution $\lambda\varepsilon^{-\lambda x}dx$. The service period has the distribution $\mu\varepsilon^{-\mu x}dx$. Arrival of customers and of service periods are mutually independent. Let: $\sigma_p=\sum_{i=1}^{p}\lambda_i$, $\sigma_r=\sigma$. We have state n if the total number of customers at the counter and queue is n; state (n, p) if one customer is at the counter and $(n-1)$ customers of priority p or higher in the queue. The equilibrium states are determined with the help of partial differential equations: If $\sigma<\mu$, then $P_k=[(\mu-\sigma)/\mu](\sigma/\mu)^k$; $P_{(0,p)}=(\mu-\sigma)/\mu$; $P_{(k,p)}=[(\mu-\sigma_p)\sigma/\mu\sigma_p](\sigma_p/\mu)^k$ $(k=1,\ 2,\ \cdots)$. If $\sigma>\mu>\sigma_p$: $P_k=0$, $P_{(0,p)}=0$, $P_{(k,p)}=[(\mu-\sigma_p)/\mu](\sigma_p/\mu)^{k-1}$. The probability that a customer of priority p will have to wait time T, conditional on the system being in state (n, p) is: $f_{(0,p)}=0$; $f_{(n,1)}=\mu^nT^{n-1}\varepsilon^{-\mu T}/(n-1)!$; $f_{(n,p)}=n(\mu/\sigma_{p-1})^{n/2}T^{-1}\exp[-(\sigma_{p-1}+\mu)T]I_n(2\sigma_{p-1}^{\frac{1}{2}}\mu^{\frac{1}{2}}T)$ $(n=1,\ 2,\ \cdots;\ p=2,\ 3,\ \cdots)$, where I_n is the modified Bessel function of order n. *G. Tintner* (Ames, Iowa).

Beckmann, Martin J. On the division of labor in teams. Metroecon. **8** (1956), 163–168.

See also: Computing Machines: Luhn. **Statistics:** Isbell and Wagner; Shiskin and Eisenpress. **Programming, Resource Allocation, Games:** Morton; Tintner.

Programming, Resource Allocation, Games

Prager, William. A generalization of Hitchcock's transportation problem. J. Math. Phys. **36** (1957), 99–106.

Kuhn, H. W. A note on Prager's transportation problem. J. Math. Phys. **36** (1957), 107–111.

In the first of these papers, Prager treats a generalized form of the Hitchcock transportation problem along lines suggested by analogy with physical field theory. The generalization consists mainly in the as-

sumption of a mixed boundary condition: at certain nodes of the transportation network "consumption rates" are specified, while at the others "prices" are stipulated. Minimum and maximum characterizations of solutions to the mixed problem are stated. In the concluding section it is remarked that the generalized problem can be formulated as a linear program, and the author's results can be obtained from general linear programming theorems.

In the second paper, Kuhn demonstrates that this is indeed the case by formulating the generalized Hitchcock problem as a linear program and showing how Prager's results follow from the weak half of the duality theorem. In his concluding comments he argues cogently that the linear programming approach is more natural for this problem, and for other problems in mathematical economics, than is the field theory approach favored by Prager. *D. R. Fulkerson* (Santa Monica, Calif.).

Prager, William. On the Caterer Problem. Management Sci. **3** (1956), 15–23.

This paper points out that the caterer problem [Jacobs, Naval Res. Logist. Quart. **1** (1954), 154–165; MR **16**, 386] can be treated as a transportation problem, if one first assumes that the total number of napkins that will be bought is known beforehand. The origins are the store and each day's hamper of soiled napkins; the destinations are the final inventory of soiled napkins and each day's requirement of fresh napkins. The author shows a method of taking advantage of the special form of the cost matrix of this problem, and also the correct method for observing the manner in which a variation in the total number of napkins to be bought from the store changes the solution. He has thus provided a method of computation alternative to that given by a combination of the ideas of Jacobs and Gaddum, Hoffman and Slokolowski [ibid. **1** (1954), 223–229; MR **16**, 843]. For the case in which the difference between fast and slow service is one day, his answer coincides exactly with that of Jacobs. It is not clear how his procedures compare with earlier ones in the general case.

E. M. L. Beale has pointed out in a letter to the editor [ibid. **4**, 110 (1957)] that it is possible to write the optimal solution to the transportation problem immediately by properly translating to the present context the idea contained in the second paper cited above. This combined approach is probably the speediest and simplest way to solve the caterer problem.

Additionally, the reviewer wishes to remark that, although this ingenious approach sheds light on the features of the problem that make an explicit solution possible, it does not shed light on the general methodology of the "Jacobs transformation". This remains as much an isolated tour de force as ever. *A. J. Hoffman.*

Bellman, Richard. On a dynamic programming approach to the caterer problem. I. Management Sci. **3** (1957), 270–278.

The Caterer Problem [W. Jacobs, Naval Res. Logist. Quart. **1** (1954), 154–165; MR **16**, 386] is a paraphrase of a practical problem concerning the number of spare engines required to assure specified operation levels of a fleet of airplanes. As this reviewer has shown [see the paper reviewed above], it is a Transportation Problem with a cost matrix of particularly simple structure. In the present paper an explicit solution of this problem is obtained by means of the dynamic programming techniques, which the author has developed in a series of papers [see, e.g., Bull. Amer. Math. Soc. **60** (1954), 503–515; MR **16**, 732]. *W. Prager* (Providence, R.I.).

Klein, Morton. Some production planning problems. Naval Res. Logist. Quart. **4** (1957), 269–286.

The most general form of the "production planning problems" considered here is: To minimize $\sum_{i=1}^{n} f_i(x_i)$, subject to some of the following constraints: (a) $\sum x_i = S$, (b) $x_i \geqq 0$, (c) $x_i \leqq c_i$, (d) $\sum_{j=1}^{i} x_j \geqq S_i$ $(i=1, \cdots, n)$, where the f_i are all convex (some remarks on f_i concave are made). Computational procedures are given for the four types of problem obtained by imposing the constraints: (i) (a) only, (ii) (a) and (b), (iii) (a) and (c) and (iv) (a) and (d). The procedure specified for case (ii) illustrates the relation between successive problems: Solve the problem first under (a) alone; set equal to zero all x which were found negative; repeat. This procedure depends on the fact that if $x_i < 0$ in (i), then $x_i = 0$ in (ii); similar relations hold with regard to the constraints added in passing from (ii) to (iii) and from (iii) to (iv). *P. Wolfe.*

Bellman, Richard. Dynamic programming and the variational solution of the Thomas-Fermi equation. J. Phys. Soc. Japan **12** (1957), 1049.

The equation in question is $u'' - u^{3/2}x^{-\frac{1}{2}} = 0$, subject to $u(0)=1$, $u(\infty)=0$; it has arisen in several applications. Recently Ikebe and Kato [J. Phys. Soc. Japan **12** (1957), 201–203] considered a variational approach for determining $u'(0)$. The author considers, along these lines, the functional $f(a, c) = \min_u \int_a^\infty (u'^2 + u^m x^{-n})dx$ subject to $u(a)=c$, $u(\infty)=0$. The solution of the Hamilton-Jacobi partial differential equation for f can easily be obtained in terms of a function of one variable satisfying an ordinary first order nonlinear differential equation. *J. Kiefer.*

Bellman, Richard. On the theory of dynamic programming — a warehousing problem. Management Sci. **2** (1956), 272–275.

The author points out that his well-known functional equation approach can be employed to solve the warehousing problem of Charnes and Cooper. An unfortunate and persistent notational error makes the formulas confusing, but the general idea is clear. The superiority of his approach over linear programming in this case is debatable, however, since Charnes and Cooper provide a very simple and straightforward algorithm. As a general principle, of course, the presentation of alternative algorithms for the same problem is at least as meritorious as the presentation of alternative proofs for the same theorem. *A. J. Hoffman* (New York, N.Y.).

★ **Morton, G. An application of dynamic programming.** Conference on linear programming, May, 1954, pp. 32–38; discussion, 39–40. Ferranti Ltd., London.

The author discusses the application of dynamic programming to some multi-stage processes arising in forestry. A longer version of the paper has since appeared, Proceedings of an International Conference on Input-Output Analysis, J. Wiley and Sons, 1956. *R. Bellman.*

Theil, H. A note on certainty equivalence in dynamic planning. Econometrica **25** (1957), 346–349.

"···Under certain conditions, the first-period action of the strategy which maximizes expected utility is identical with that of the strategy which neglects the uncertainty problem by maximizing utility under the condition that

all uncertain elements are equal to their mean values." More specifically, assume that: (1) The "utility" $w(x, y)$ is a quadratic function of two sequences of length T: $x=\{x_t\}$, the time-sequence of "instruments", and $y=\{y_t\}$, the time-sequence of "noncontrolled variables"; where $x_t=\{x_{it}\}$, $y_t=\{y_{jt}\}$ are themselves vectors, $i=1, \cdots, m$; $j=1, \cdots, n$; (2) there exists a fixed $T\times T$ matrix $R=[R_{tu}]$ with $R_{tu}=0$ for $u>t$, and a random sequence $s=\{s_t\}$ of length T with $s_t=\{s_{jt}\}$, $(j=1, \cdots, n)$, such that s is independent of x, and that the column vector $y'=x'R+s$. [Condition (2) ensures that the policy-maker who knows R knows, at time $u>t$, not only x_t, y_t but also s_t, the "disturbance".] A strategy is a sequence $f=(f_1, \cdots, f_T)$ of functions, the argument of f_t being the sequence $s^t=(0, s_1, s_2, \cdots, s_{t-1})$, and $f_t(s^t)=x_t$. Let f^* be the optimal strategy; that is, $x_t^*=f_t^*(s^t)$, $(t=1, \cdots, T)$, maximizes the expected utility Ew. Let $\bar{x}$ be that value of x which maximizes w itself, when s is replaced by its expectation. (1) and (2) imply that the first subvector $\bar{x}_1$ of $\bar{x}$ is equal to the first subvector $f_1^*(0)$ of $f^*(s^t)$.

This result is a vector generalization (with a simpler proof) of that of H. A. Simon [Econometrica **24** (1956), 74–81; MR **17**, 1104]. It is also more general than H. Theil's own result for the static case $T=1$ [Weltwirtschaftliches Arch. **72** (1954), 60–83], except that in the static case, w needs to be quadratic in y only, and y need not be linearly related to x. [For some applications of the Simon-Theil result see also C. C. Holt, F. Modigliani and H. A. Simon, Management Sci. **2** (1955), 1–30.]

J. Marschak (New Haven, Conn.).

Bellman, Richard. Dynamic programming and a new formalism in the calculus of variations. I. Riv. Mat. Univ. Parma **6** (1955), 193–213.

Tintner, Gerhard. Game theory, linear programming and input-output analysis. Z. Nationalökonomie **17** (1957), 1–38.

This is an expository paper on the subjects named in the title. *D. Gale* (Santa Monica, Calif.).

See also: Numerical Methods: Wagner; Fulkerson and Dantzig; Dreyfus.

Biology and Sociology

Bailey, Norman T. J. On estimating the latent and infectious periods of measles. I. Biometrika **43** (1956), 15–22.

Infection is assumed to be followed by a latent period, with the distribution $N(m, \sigma^2)$, and a subsequent infectious period of constant length a. Infection of a second susceptible during this time follows a Poisson process with parameter λ. Details are given of the maximum likelihood solution for families of two susceptibles, in which the observed data is the numbers of families with 1 and 2 cases, and the distribution of the interval between cases in the latter group. The effects of variations in the parameters are discussed, as is the allowance for misclassification of families of two cases into those infected from a common source and those with one external and one internal infection. *P. Armitage* (London).

Bailey, Norman T. J. On estimating the latent and infectious periods of measles. II. Biometrika **43** (1956), 322–331.

With the same model as in Part I (reviewed above), the maximum likelihood solution is given for families of three susceptibles. *P. Armitage* (London).

Bailey, Norman T. J. Significance tests for a variable chance of infection in chain-binomial theory. Biometrika **43** (1956), 332–336.

In chain-binomial theory, as an alternative to assuming a constant probability parameter p, this may be assumed to be a B-variate, with parameters x, y. A test, based on the derivative of the log likelihood, is described for the null hypothesis that p is constant, i.e., x and y are infinite. *P. Armitage* (London).

Marchand, Henri. Essai d'étude mathématique d'une forme d'épidémie. Ann. Univ. Lyon. Sect. A. (3) **19** (1956), 13–46.

In the first part, the author studies a deterministic epidemic model in which a population of N individuals is composed at time t of $x(t)$ uninfected, $y(t)$ infective and $N-x-y$ isolated individuals. The rate of occurrence of new infections is proportional to xy, and an infected individual is infective for constant time T, after which he is isolated. The author examines the behaviour of the total epidemic size as a function of the parameters, and obtains a limiting expression (as $T\to 0$) for $x(t)$. In the second part a stochastic model is examined, in which $x(t)$ is a continuous variate. A threshold theorem, analogous to one appearing in the deterministic model, is obtained under the limiting condition $T\to 0$. This gives necessary and sufficient conditions under which the derivative of the stochastic mean of $x(t)$ has a maximum at some $t>0$. *P. Armitage* (London).

Kemeny, John G.; and Snell, J. Laurie. Markov processes in learning theory. Psychometrika **22** (1957), 221–230.

A learning process proposed by Estes and Burke [Psychol. Rev. **60** (1953), 276–286] is treated as a Markov chain and is shown to lead, in a limiting form, to a model proposed by Bush and Mosteller [Stochastic models for learning, Wiley, New York, 1955; MR **16**, 1136]. The limiting distributions are derived for both models. The basic process can be described as follows. Two urns, A and B, contain in all n balls. Urn A is chosen with probability p, urn B with probability $1-p$, and each ball in the chosen urn is placed in the other urn with probability θ. *P. Armitage* (London).

Guttman, Louis. Some necessary conditions for common-factor analysis. Psychometrika **19** (1954), 149–161.

★ **Guttman, Louis. A new approach to factor analysis: the radex.** Mathematical thinking in the social sciences, pp. 258–348, 430–433. The Free Press, Glencoe, Ill., 1954. $10.00.

Information and Communication Theory

★ **Rapoport, Anatol. Random nets with transitivity bias.** Proceedings of the symposium on information networks, New York, April, 1954, pp. 187–197. Polytechnic Institute of Brooklyn, Brooklyn, N. Y., 1955.

See also: Statistics: Sakaguchi.

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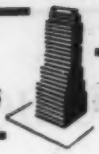

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